Homework 12
Linear Algebra
(MATH 442), Spring, 2026

Due: 05/06/2026

Section 7.D - Isometries, Unitary Operators, and QR Factorization

Problem 1. Suppose V is an inner product space and $S \in \mathcal{L}(V)$ is a unitary operator. Let $\mathbb{B}$ denote the closed unit ball in V; i.e., the set $\mathbb{B} = \{v \in V : \|v\| \le 1\}$. Show that $S(\mathbb{B}) = \mathbb{B}$, where $S(\mathbb{B}) = \{Sv : v \in \mathbb{B}\}$

Problem 2. Suppose V is an inner product space over $\mathbb{C}$ and $T \in \mathcal{L}(V)$. Suppose that every eigenvalue λ of T satisfies $|\lambda| = 1$ and suppose that $\|Tv\| \le \|v\|$ for every $v \in V$. Prove that T is unitary.

Section 7.E - Singular Value Decomposition

Problem 3. Suppose V and W are inner product spaces and $T \in \mathcal{L}(V, W)$. Prove that $s > 0$ is a singular value of T if and only if there exist nonzero vectors $v \in V$ and $w \in W$ such that $Tv = sw$ and $T^*w = sv$.

Problem 4. Suppose Suppose that $T \in \mathcal{L}(V)$ is self-adjoint or that $\mathbb{F} = \mathbb{C}$ and $T \in \mathcal{L}(V)$ is normal. Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of T, each included in this list as many times as the dimension of the corresponding eigenspace. Show that the singular values of T are $|\lambda_1|, \ldots, |\lambda_n|$, after these numbers are sorted into decreasing order.