Homework 6
Linear Algebra
(MATH 442), Spring, 2026

Due: 03/20/2026

Section 5.A - Invariant Subspaces

Problem 1. Suppose $T \in \mathcal{L}(W)$ and suppose $S \in \mathcal{L}(W)$ is invertible.

(a) Prove that T and $S^{-1}TS$ have the same eigenvalues.

(b) Prove that $S^{-1}w$ is an eigenvector of $S^{-1}TS$ for every eigenvector w of T.

Problem 2. Suppose V is a finite-dimensional vector space, $T \in \mathcal{L}(V)$, and $\lambda \in \mathbb{F}$. Show that λ is an eigenvalue of T if and only if λ is an eigenvalue of the dual operator $T' \in \mathcal{L}(V')$.

Section 5.B (Part 1) - Eigenvalues and the Minimal Polynomial

Problem 3. Suppose W is a finite-dimensional vector space over $\mathbb{C}$, $T \in \mathcal{L}(W)$, $p \in \mathscr{P}(\mathbb{C})$, and $\alpha \in \mathbb{C}$. Prove that α is an eigenvalue of $p(T)$ if and only if $\alpha = p(\lambda)$ for some eigenvalue λ of T.

Problem 4. Suppose V is finite-dimensional, $T \in \mathcal{L}(V)$, and $v \in V$. Prove that

$$\operatorname{span}(v, Tv, \ldots, T^m v) = \operatorname{span}(v, Tv, \ldots, T^{\dim V - 1} v)$$

for all integers $m \geq \dim V - 1$.

Problem 5. Suppose V is finite-dimensional, $T \in \mathcal{L}(V)$. Let $\mathcal{E}$ be the subspace of $\mathcal{L}(V)$ defined by

$$\mathcal{E} = \{q(T) : q \in \mathcal{P}(\mathbb{F})\}.$$

Prove that dim $\mathcal{E}$ equals the degree of the minimal polynomial of T.