Homework 5
Linear Algebra
(MATH 442), Spring, 2026

Due: 02/27/2026

Section 3.E - Products and Quotients of Vector Spaces

Problem 1. Suppose v, x are vectors in V and that U, W are subspaces of V such that $v + \mathrm{U} = x + \mathrm{W}$. Prove that $\mathrm{U} = \mathrm{W}$.

Problem 2. A subspace $\mathrm{U} \leq \mathrm{V}$ is said to be *of codimension m* if $\dim\big(\mathrm{V}/\mathrm{U}\big) = m$. In this problem, you will show that a subspaces of a vector space is of codimension 1 if and only if it is the null space of a linear functional.

(a) Suppose $\phi \in \mathcal{L}(\mathrm{V}, \mathbb{F})$ and $\phi \neq 0$. Prove that $\dim\Big(\mathrm{V}/\mathrm{null}(\phi)\Big) = 1$.

(b) Suppose $\mathrm{U} \leq \mathrm{V}$ is such that $\dim\big(\mathrm{V}/\mathrm{U}\big) = 1$. Prove that there exists $\phi \in \mathcal{L}(\mathrm{V}, \mathbb{F})$ such that $\mathrm{null}(\phi) = \mathrm{U}$.

Section 3.F - Duality

Problem 3. The *double dual space* of V, denoted V'', is the dual space of V', i.e. $\mathrm{V}'' = (\mathrm{V}')'$. Define $\Lambda : \mathrm{V} \to \mathrm{V}''$ by $(\Lambda v)(\phi) = \phi(v)$ for $v \in \mathrm{V}$ and $\phi \in \mathrm{V}'$.

1. Show that $\Lambda \in \mathcal{L}(\mathrm{V}, \mathrm{V}'')$.
2. Show that if $\mathrm{T} \in \mathcal{L}(\mathrm{V})$, then $\mathrm{T}'' \circ \Lambda = \Lambda \circ \mathrm{T}$, where $\mathrm{T}'' = (\mathrm{T}')'$.
3. Show that if V is finite-dimensional, then Λ is an isomorphism from V onto V''.

Problem 4. Suppose $v_1, \ldots, v_n$ is a basis of V and $\phi_1, \ldots, \phi_n$ is the corresponding dual basis of V'. Suppose $\psi \in \mathrm{V}'$. Prove that $\psi = \psi(v_1)\phi_1 + \ldots + \psi(v_n)\phi_n$.