Homework 3
Linear Algebra
(MATH 442), Spring, 2026

Due: 02/13/2026

Section 2.C - Dimension

Problem 1. Suppose $v_1, \ldots, v_m$ is linearly independent in V and $w \in V$. Prove that

$$\dim\Big(\operatorname{span}\Big(v_1 + w, \ldots, v_m + w\Big)\Big) \geq m - 1.$$

Problem 2. Suppose V is finite-dimensional and U is a subspace of V with $U \neq V$. Let $\dim\Big(V\Big) = n$ and $\dim\Big(U\Big) = m$. Prove that there exist $n - m$ subspaces of V, each with dimension $n - 1$, whose intersection equals U.

Section 3.A - Vector Space of Linear Maps

Problem 3. Suppose that $T \in \mathcal{L}(\mathbb{F}^n, \mathbb{F}^m)$. Show that there exist scalars $A_{j,k} \in \mathbb{F}$ for $j = 1, \ldots, m$ and $k = 1, \ldots, n$ such that

$$T(x_1, \ldots, x_n) = \Big((A_{1,1}x_1 + \ldots + A_{1,n}x_n), \ldots, (A_{m,1}x_1 + \ldots + A_{m,n}x_n)\Big).$$

Problem 4. Suppose that $T \in \mathcal{L}(V, W)$ and that $v_1, \ldots, v_m$ is a list of vectors in V such that $Tv_1, \ldots, Tv_m$ is a linearly independent list in W. Prove that $v_1, \ldots, v_m$ is linearly independent.

Section 3.B - Null Spaces and Ranges

Problem 5. Prove that there does not exist a linear map $T : \mathbb{R}^5 \to \mathbb{R}^5$ such that $\operatorname{range} T = \operatorname{null} T$.

Problem 6. Suppose $\phi \in \mathcal{L}(V, \mathbb{F})$ and that $u \notin \operatorname{null}\phi$. Prove that $V = \operatorname{null}\phi \oplus \Big\{au : a \in \mathbb{F}\Big\}$.