7C Positive Operators

7.34 definition: *positive operator*

An operator $T \in \mathcal{L}(V)$ is called *positive* if T is self-adjoint and

$$\langle Tv, v \rangle \geq 0$$

for all $v \in V$.

If V is a complex vector space, then the requirement that T be self-adjoint can be dropped from the definition above (by 7.14).

7.35 example: *positive operators*

(a) Let $T \in \mathcal{L}(\mathbf{F}^2)$ be the operator whose matrix (using the standard basis) is $\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$. Then T is self-adjoint and $\langle T(w,z), (w,z) \rangle = 2|w|^2 - 2\operatorname{Re}(w\bar{z}) + |z|^2 = |w-z|^2 + |w|^2 \geq 0$ for all $(w,z) \in \mathbf{F}^2$. Thus T is a positive operator.

(b) If U is a subspace of V, then the orthogonal projection P_U is a positive operator, as you should verify.

(c) If $T \in \mathcal{L}(V)$ is self-adjoint and $b, c \in \mathbf{R}$ are such that $b^2 < 4c$, then $T^2 + bT + cI$ is a positive operator, as shown by the proof of 7.26.

7.36 definition: *square root*

An operator R is called a *square root* of an operator T if $R^2 = T$.

7.37 example: *square root of an operator*

If $T \in \mathcal{L}(\mathbf{F}^3)$ is defined by $T(z_1, z_2, z_3) = (z_3, 0, 0)$, then the operator $R \in \mathcal{L}(\mathbf{F}^3)$ defined by $R(z_1, z_2, z_3) = (z_2, z_3, 0)$ is a square root of T because $R^2 = T$, as you can verify.

The characterizations of the positive operators in the next result correspond to characterizations of the nonnegative numbers among $\mathbf{C}$. Specifically, a number $z \in \mathbf{C}$ is nonnegative if and only if it has a nonnegative square root, corresponding to condition (d). Also, z is nonnegative if and only if it has a real square root, corresponding to condition (e). Finally, z is nonnegative if and only if there exists $w \in \mathbf{C}$ such that $z = \bar{w}w$, corresponding to condition (f). See Exercise 20 for another condition that is equivalent to being a positive operator.

Because positive operators correspond to nonnegative numbers, better terminology would use the term nonnegative operators. However, operator theorists consistently call these positive operators, so we follow that custom. Some mathematicians use the term ***positive semidefinite operator****, which means the same as positive operator.*

7.38 *characterization of positive operators*

Let $T \in \mathcal{L}(V)$. Then the following are equivalent.

(a) T is a positive operator.

(b) T is self-adjoint and all eigenvalues of T are nonnegative.

(c) With respect to some orthonormal basis of V, the matrix of T is a diagonal matrix with only nonnegative numbers on the diagonal.

(d) T has a positive square root.

(e) T has a self-adjoint square root.

(f) $T = R^*R$ for some $R \in \mathcal{L}(V)$.

Proof We will prove that (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d) $\Rightarrow$ (e) $\Rightarrow$ (f) $\Rightarrow$ (a).

First suppose (a) holds, so that T is positive, which implies that T is self-adjoint (by definition of positive operator). To prove the other condition in (b), suppose λ is an eigenvalue of T. Let v be an eigenvector of T corresponding to λ. Then

$$0 \leq \langle Tv, v \rangle = \langle \lambda v, v \rangle = \lambda \langle v, v \rangle.$$

Thus λ is a nonnegative number. Hence (b) holds, showing that (a) implies (b).

Now suppose (b) holds, so that T is self-adjoint and all eigenvalues of T are nonnegative. By the spectral theorem (7.29 and 7.31), there is an orthonormal basis $e_1, \ldots, e_n$ of V consisting of eigenvectors of T. Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of T corresponding to $e_1, \ldots, e_n$; thus each λ_k is a nonnegative number. The matrix of T with respect to $e_1, \ldots, e_n$ is the diagonal matrix with $\lambda_1, \ldots, \lambda_n$ on the diagonal, which shows that (b) implies (c).

Now suppose (c) holds. Suppose $e_1, \ldots, e_n$ is an orthonormal basis of V such that the matrix of T with respect to this basis is a diagonal matrix with nonnegative numbers $\lambda_1, \ldots, \lambda_n$ on the diagonal. The linear map lemma (3.4) implies that there exists $R \in \mathcal{L}(V)$ such that

$$Re_k = \sqrt{\lambda_k} e_k$$

for each $k = 1, \ldots, n$. As you should verify, R is a positive operator. Furthermore, $R^2 e_k = \lambda_k e_k = Te_k$ for each k, which implies that $R^2 = T$. Thus R is a positive square root of T. Hence (d) holds, which shows that (c) implies (d).

Every positive operator is self-adjoint (by definition of positive operator). Thus (d) implies (e).

Now suppose (e) holds, meaning that there exists a self-adjoint operator R on V such that $T = R^2$. Then $T = R^*R$ (because $R^* = R$). Hence (e) implies (f).

Finally, suppose (f) holds. Let $R \in \mathcal{L}(V)$ be such that $T = R^*R$. Then $T^* = (R^*R)^* = R^*(R^*)^* = R^*R = T$. Hence T is self-adjoint. To complete the proof that (a) holds, note that

$$\langle Tv, v \rangle = \langle R^*Rv, v \rangle = \langle Rv, Rv \rangle \geq 0$$

for every $v \in V$. Thus T is positive, showing that (f) implies (a). ■

Every nonnegative number has a unique nonnegative square root. The next result shows that positive operators enjoy a similar property.

7.39 *each positive operator has only one positive square root*

Every positive operator on V has a unique positive square root.

A positive operator can have infinitely many square roots (although only one of them can be positive). For example, the identity operator on V has infinitely many square roots if $\dim V > 1$.

Proof Suppose $T \in \mathcal{L}(V)$ is positive. Suppose $v \in V$ is an eigenvector of T. Hence there exists a real number $\lambda \geq 0$ such that $Tv = \lambda v$.

Let R be a positive square root of T. We will prove that $Rv = \sqrt{\lambda}v$. This will imply that the behavior of R on the eigenvectors of T is uniquely determined. Because there is a basis of V consisting of eigenvectors of T (by the spectral theorem), this will imply that R is uniquely determined.

To prove that $Rv = \sqrt{\lambda}v$, note that the spectral theorem asserts that there is an orthonormal basis $e_1, \dots, e_n$ of V consisting of eigenvectors of R. Because R is a positive operator, all its eigenvalues are nonnegative. Thus there exist nonnegative numbers $\lambda_1, \dots, \lambda_n$ such that $Re_k = \sqrt{\lambda_k}e_k$ for each $k = 1, \dots, n$.

Because $e_1, \dots, e_n$ is a basis of V, we can write

$$v = a_1 e_1 + \cdots + a_n e_n$$

for some numbers $a_1, \dots, a_n \in \mathbf{F}$. Thus

$$Rv = a_1\sqrt{\lambda_1}e_1 + \cdots + a_n\sqrt{\lambda_n}e_n.$$

Hence

$$\lambda v = Tv = R^2 v = a_1\lambda_1 e_1 + \cdots + a_n\lambda_n e_n.$$

The equation above implies that

$$a_1\lambda e_1 + \cdots + a_n\lambda e_n = a_1\lambda_1 e_1 + \cdots + a_n\lambda_n e_n.$$

Thus $a_k(\lambda - \lambda_k) = 0$ for each $k = 1, \dots, n$. Hence

$$v = \sum_{\{k \,:\, \lambda_k = \lambda\}} a_k e_k.$$

Thus

$$Rv = \sum_{\{k \,:\, \lambda_k = \lambda\}} a_k\sqrt{\lambda}e_k = \sqrt{\lambda}v,$$

as desired. ∎

The notation defined below makes sense thanks to the result above.

7.40 notation: $\sqrt{T}$

For T a positive operator, $\sqrt{T}$ denotes the unique positive square root of T.

7.41 example: *square root of positive operators*

Define operators S, T on $\mathbf{R}^2$ (with the usual Euclidean inner product) by

$$S(x,y) = (x, 2y) \quad \text{and} \quad T(x,y) = (x+y, x+y).$$

Then with respect to the standard basis of $\mathbf{R}^2$ we have

7.42
$$\mathcal{M}(S) = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \quad \text{and} \quad \mathcal{M}(T) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Each of these matrices equals its transpose; thus S and T are self-adjoint.

If $(x,y) \in \mathbf{R}^2$, then

$$\langle S(x,y), (x,y) \rangle = x^2 + 2y^2 \geq 0$$

and

$$\langle T(x,y), (x,y) \rangle = x^2 + 2xy + y^2 = (x+y)^2 \geq 0.$$

Thus S and T are positive operators.

The standard basis of $\mathbf{R}^2$ is an orthonormal basis consisting of eigenvectors of S. Note that

$$\left(\tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}\right), \left(\tfrac{1}{\sqrt{2}}, -\tfrac{1}{\sqrt{2}}\right)$$

is an orthonormal basis of eigenvectors of T, with eigenvalue 2 for the first eigenvector and eigenvalue 0 for the second eigenvector. Thus $\sqrt{T}$ has the same eigenvectors, with eigenvalues $\sqrt{2}$ and 0.

You can verify that

$$\mathcal{M}\left(\sqrt{S}\right) = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{2} \end{pmatrix} \quad \text{and} \quad \mathcal{M}\left(\sqrt{T}\right) = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

with respect to the standard basis by showing that the squares of the matrices above are the matrices in 7.42 and that each matrix above is the matrix of a positive operator.

The statement of the next result does not involve a square root, but the clean proof makes nice use of the square root of a positive operator.

7.43 *T positive and $\langle Tv, v \rangle = 0 \implies Tv = 0$*

Suppose T is a positive operator on V and $v \in V$ is such that $\langle Tv, v \rangle = 0$. Then $Tv = 0$.

Proof We have

$$0 = \langle Tv, v \rangle = \left\langle \sqrt{T}\sqrt{T}v, v \right\rangle = \left\langle \sqrt{T}v, \sqrt{T}v \right\rangle = \left\| \sqrt{T}v \right\|^2.$$

Hence $\sqrt{T}v = 0$. Thus $Tv = \sqrt{T}\left(\sqrt{T}v\right) = 0$, as desired. ■

Exercises 7C

1 Suppose $T \in \mathcal{L}(V)$. Prove that if both T and $-T$ are positive operators, then $T = 0$.

2 Suppose $T \in \mathcal{L}(\mathbf{F}^4)$ is the operator whose matrix (with respect to the standard basis) is

$$\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}.$$

Show that T is an invertible positive operator.

3 Suppose n is a positive integer and $T \in \mathcal{L}(\mathbf{F}^n)$ is the operator whose matrix (with respect to the standard basis) consists of all 1's. Show that T is a positive operator.

4 Suppose n is an integer with $n > 1$. Show that there exists an n-by-n matrix A such that all of the entries of A are positive numbers and $A = A^*$, but the operator on $\mathbf{F}^n$ whose matrix (with respect to the standard basis) equals A is not a positive operator.

5 Suppose $T \in \mathcal{L}(V)$ is self-adjoint. Prove that T is a positive operator if and only if for every orthonormal basis $e_1, \ldots, e_n$ of V, all entries on the diagonal of $\mathcal{M}(T, (e_1, \ldots, e_n))$ are nonnegative numbers.

6 Prove that the sum of two positive operators on V is a positive operator.

7 Suppose $S \in \mathcal{L}(V)$ is an invertible positive operator and $T \in \mathcal{L}(V)$ is a positive operator. Prove that $S + T$ is invertible.

8 Suppose $T \in \mathcal{L}(V)$. Prove that T is a positive operator if and only if the pseudoinverse $T^\dagger$ is a positive operator.

9 Suppose $T \in \mathcal{L}(V)$ is a positive operator and $S \in \mathcal{L}(W, V)$. Prove that S^*TS is a positive operator on W.

10 Suppose T is a positive operator on V. Suppose $v, w \in V$ are such that

$$Tv = w \quad \text{and} \quad Tw = v.$$

Prove that $v = w$.

11 Suppose T is a positive operator on V and U is a subspace of V invariant under T. Prove that $T|_U \in \mathcal{L}(U)$ is a positive operator on U.

12 Suppose $T \in \mathcal{L}(V)$ is a positive operator. Prove that T^k is a positive operator for every positive integer k.

13 Suppose $T \in \mathcal{L}(V)$ is self-adjoint and $\alpha \in \mathbf{R}$.

(a) Prove that $T - \alpha I$ is a positive operator if and only if α is less than or equal to every eigenvalue of T.

(b) Prove that $\alpha I - T$ is a positive operator if and only if α is greater than or equal to every eigenvalue of T.

14 Suppose T is a positive operator on V and $v_1, \ldots, v_m \in V$. Prove that

$$\sum_{j=1}^{m} \sum_{k=1}^{m} \langle T v_k, v_j \rangle \geq 0.$$

15 Suppose $T \in \mathcal{L}(V)$ is self-adjoint. Prove that there exist positive operators $A, B \in \mathcal{L}(V)$ such that

$$T = A - B \quad \text{and} \quad \sqrt{T^* T} = A + B \quad \text{and} \quad AB = BA = 0.$$

16 Suppose T is a positive operator on V. Prove that

$$\operatorname{null} \sqrt{T} = \operatorname{null} T \quad \text{and} \quad \operatorname{range} \sqrt{T} = \operatorname{range} T.$$

17 Suppose that $T \in \mathcal{L}(V)$ is a positive operator. Prove that there exists a polynomial p with real coefficients such that $\sqrt{T} = p(T)$.

18 Suppose S and T are positive operators on V. Prove that ST is a positive operator if and only if S and T commute.

19 Show that the identity operator on $\mathbf{F}^2$ has infinitely many self-adjoint square roots.

20 Suppose $T \in \mathcal{L}(V)$ and $e_1, \ldots, e_n$ is an orthonormal basis of V. Prove that T is a positive operator if and only if there exist $v_1, \ldots, v_n \in V$ such that

$$\langle T e_k, e_j \rangle = \langle v_k, v_j \rangle$$

for all $j, k = 1, \ldots, n$.

The numbers $\{\langle T e_k, e_j \rangle\}_{j,k=1,\ldots,n}$ are the entries in the matrix of T with respect to the orthonormal basis $e_1, \ldots, e_n$.

21 Suppose n is a positive integer. The n-by-n *Hilbert matrix* is the n-by-n matrix whose entry in row j, column k is $\frac{1}{j+k-1}$. Suppose $T \in \mathcal{L}(V)$ is an operator whose matrix with respect to some orthonormal basis of V is the n-by-n Hilbert matrix. Prove that T is a positive invertible operator.

Example: The 4-by-4 Hilbert matrix is

$$\begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} \end{pmatrix}.$$

22 Suppose $T \in \mathcal{L}(V)$ is a positive operator and $u \in V$ is such that $\|u\| = 1$ and $\|Tu\| \geq \|Tv\|$ for all $v \in V$ with $\|v\| = 1$. Show that u is an eigenvector of T corresponding to the largest eigenvalue of T.

23 For $T \in \mathcal{L}(V)$ and $u, v \in V$, define $\langle u, v \rangle_T$ by $\langle u, v \rangle_T = \langle Tu, v \rangle$.

(a) Suppose $T \in \mathcal{L}(V)$. Prove that $\langle \cdot, \cdot \rangle_T$ is an inner product on V if and only if T is an invertible positive operator (with respect to the original inner product $\langle \cdot, \cdot \rangle$).

(b) Prove that every inner product on V is of the form $\langle \cdot, \cdot \rangle_T$ for some positive invertible operator $T \in \mathcal{L}(V)$.

24 Suppose S and T are positive operators on V. Prove that

$$\operatorname{null}(S + T) = \operatorname{null} S \cap \operatorname{null} T.$$

25 Let T be the second derivative operator in Exercise 31(b) in Section 7A. Show that $-T$ is a positive operator.