

1B Definition of Vector Space

The motivation for the definition of a vector space comes from properties of addition and scalar multiplication in $\mathbf{F}^n$: Addition is commutative, associative, and has an identity. Every element has an additive inverse. Scalar multiplication is associative. Scalar multiplication by 1 acts as expected. Addition and scalar multiplication are connected by distributive properties.

We will define a vector space to be a set V with an addition and a scalar multiplication on V that satisfy the properties in the paragraph above.

1.19 definition: *addition, scalar multiplication*

- An *addition* on a set V is a function that assigns an element $u + v \in V$ to each pair of elements $u, v \in V$.
- A *scalar multiplication* on a set V is a function that assigns an element $\lambda v \in V$ to each $\lambda \in \mathbf{F}$ and each $v \in V$.

Now we are ready to give the formal definition of a vector space.

1.20 definition: *vector space*

A *vector space* is a set V along with an addition on V and a scalar multiplication on V such that the following properties hold.

commutativity

$$u + v = v + u \text{ for all } u, v \in V.$$

associativity

$$(u + v) + w = u + (v + w) \text{ and } (ab)v = a(bv) \text{ for all } u, v, w \in V \text{ and for all } a, b \in \mathbf{F}.$$

additive identity

$$\text{There exists an element } 0 \in V \text{ such that } v + 0 = v \text{ for all } v \in V.$$

additive inverse

$$\text{For every } v \in V, \text{ there exists } w \in V \text{ such that } v + w = 0.$$

multiplicative identity

$$1v = v \text{ for all } v \in V.$$

distributive properties

$$a(u + v) = au + av \text{ and } (a + b)v = av + bv \text{ for all } a, b \in \mathbf{F} \text{ and all } u, v \in V.$$

The following geometric language sometimes aids our intuition.

1.21 definition: *vector, point*

Elements of a vector space are called *vectors* or *points*.

The scalar multiplication in a vector space depends on $\mathbf{F}$. Thus when we need to be precise, we will say that V is a *vector space over $\mathbf{F}$* instead of saying simply that V is a vector space. For example, $\mathbf{R}^n$ is a vector space over $\mathbf{R}$, and $\mathbf{C}^n$ is a vector space over $\mathbf{C}$.

1.22 definition: *real vector space, complex vector space*

- A vector space over $\mathbf{R}$ is called a *real vector space*.
- A vector space over $\mathbf{C}$ is called a *complex vector space*.

Usually the choice of $\mathbf{F}$ is either clear from the context or irrelevant. Thus we often assume that $\mathbf{F}$ is lurking in the background without specifically mentioning it.

With the usual operations of addition and scalar multiplication, $\mathbf{F}^n$ is a vector space over $\mathbf{F}$, as you should verify. The example of $\mathbf{F}^n$ motivated our definition of vector space.

The simplest vector space is $\{0\}$, which contains only one point.

1.23 example: $\mathbf{F}^\infty$

$\mathbf{F}^\infty$ is defined to be the set of all sequences of elements of $\mathbf{F}$:

$$\mathbf{F}^\infty = \{(x_1, x_2, \dots) : x_k \in \mathbf{F} \text{ for } k = 1, 2, \dots\}.$$

Addition and scalar multiplication on $\mathbf{F}^\infty$ are defined as expected:

$$\begin{aligned}(x_1, x_2, \dots) + (y_1, y_2, \dots) &= (x_1 + y_1, x_2 + y_2, \dots), \\ \lambda(x_1, x_2, \dots) &= (\lambda x_1, \lambda x_2, \dots).\end{aligned}$$

With these definitions, $\mathbf{F}^\infty$ becomes a vector space over $\mathbf{F}$, as you should verify. The additive identity in this vector space is the sequence of all 0's.

Our next example of a vector space involves a set of functions.

1.24 notation: $\mathbf{F}^S$

- If S is a set, then $\mathbf{F}^S$ denotes the set of functions from S to $\mathbf{F}$.
- For $f, g \in \mathbf{F}^S$, the *sum* $f + g \in \mathbf{F}^S$ is the function defined by

$$(f + g)(x) = f(x) + g(x)$$

for all $x \in S$.

- For $\lambda \in \mathbf{F}$ and $f \in \mathbf{F}^S$, the *product* $\lambda f \in \mathbf{F}^S$ is the function defined by

$$(\lambda f)(x) = \lambda f(x)$$

for all $x \in S$.

As an example of the notation above, if S is the interval $[0, 1]$ and $\mathbf{F} = \mathbf{R}$, then $\mathbf{R}^{[0,1]}$ is the set of real-valued functions on the interval $[0, 1]$.

You should verify all three bullet points in the next example.

1.25 example: $\mathbf{F}^S$ is a vector space

- If S is a nonempty set, then $\mathbf{F}^S$ (with the operations of addition and scalar multiplication as defined above) is a vector space over $\mathbf{F}$.
- The additive identity of $\mathbf{F}^S$ is the function $0 : S \rightarrow \mathbf{F}$ defined by

$$0(x) = 0$$

for all $x \in S$.

- For $f \in \mathbf{F}^S$, the additive inverse of f is the function $-f : S \rightarrow \mathbf{F}$ defined by

$$(-f)(x) = -f(x)$$

for all $x \in S$.

The vector space $\mathbf{F}^n$ is a special case of the vector space $\mathbf{F}^S$ because each $(x_1, \dots, x_n) \in \mathbf{F}^n$ can be thought of as a function x from the set $\{1, 2, \dots, n\}$ to $\mathbf{F}$ by writing $x(k)$ instead of x_k for the k^{th} coordinate of $(x_1, \dots, x_n)$. In other words, we can think of $\mathbf{F}^n$ as $\mathbf{F}^{\{1, 2, \dots, n\}}$. Similarly, we can think of $\mathbf{F}^\infty$ as $\mathbf{F}^{\{1, 2, \dots\}}$.

The elements of the vector space $\mathbf{R}^{[0,1]}$ are real-valued functions on $[0, 1]$, not lists. In general, a vector space is an abstract entity whose elements might be lists, functions, or weird objects.

Soon we will see further examples of vector spaces, but first we need to develop some of the elementary properties of vector spaces.

The definition of a vector space requires it to have an additive identity. The next result states that this identity is unique.

1.26 unique additive identity

A vector space has a unique additive identity.

Proof Suppose 0 and $0'$ are both additive identities for some vector space V . Then

$$0' = 0' + 0 = 0 + 0' = 0,$$

where the first equality holds because 0 is an additive identity, the second equality comes from commutativity, and the third equality holds because $0'$ is an additive identity. Thus $0' = 0$, proving that V has only one additive identity. ■

Each element v in a vector space has an additive inverse, an element w in the vector space such that $v + w = 0$. The next result shows that each element in a vector space has only one additive inverse.

1.27 *unique additive inverse*

Every element in a vector space has a unique additive inverse.

Proof Suppose V is a vector space. Let $v \in V$. Suppose w and w' are additive inverses of v . Then

$$w = w + 0 = w + (v + w') = (w + v) + w' = 0 + w' = w'.$$

Thus $w = w'$, as desired. ■

Because additive inverses are unique, the following notation now makes sense.

1.28 notation: $-v$, $w - v$

Let $v, w \in V$. Then

- $-v$ denotes the additive inverse of v ;
- $w - v$ is defined to be $w + (-v)$.

Almost all results in this book involve some vector space. To avoid having to restate frequently that V is a vector space, we now make the necessary declaration once and for all.

1.29 notation: V

For the rest of this book, V denotes a vector space over $\mathbf{F}$.

In the next result, 0 denotes a scalar (the number $0 \in \mathbf{F}$) on the left side of the equation and a vector (the additive identity of V) on the right side of the equation.

1.30 *the number 0 times a vector*

$0v = 0$ for every $v \in V$.

Proof For $v \in V$, we have

$$0v = (0 + 0)v = 0v + 0v.$$

Adding the additive inverse of $0v$ to both sides of the equation above gives $0 = 0v$, as desired. ■

The result in 1.30 involves the additive identity of V and scalar multiplication. The only part of the definition of a vector space that connects vector addition and scalar multiplication is the distributive property. Thus the distributive property must be used in the proof of 1.30.

In the next result, 0 denotes the additive identity of V . Although their proofs are similar, 1.30 and 1.31 are not identical. More precisely, 1.30 states that the product of the scalar 0 and any vector equals the vector 0 , whereas 1.31 states that the product of any scalar and the vector 0 equals the vector 0 .

1.31 *a number times the vector 0*
 $a0 = 0$ for every $a \in \mathbf{F}$.

Proof For $a \in \mathbf{F}$, we have

$$a0 = a(0 + 0) = a0 + a0.$$

Adding the additive inverse of $a0$ to both sides of the equation above gives $0 = a0$, as desired. ■

Now we show that if an element of V is multiplied by the scalar -1 , then the result is the additive inverse of the element of V .

1.32 *the number -1 times a vector*
 $(-1)v = -v$ for every $v \in V$.

Proof For $v \in V$, we have

$$v + (-1)v = 1v + (-1)v = (1 + (-1))v = 0v = 0.$$

This equation says that $(-1)v$, when added to v , gives 0. Thus $(-1)v$ is the additive inverse of v , as desired. ■

Exercises 1B

- 1 Prove that $-(-v) = v$ for every $v \in V$.
- 2 Suppose $a \in \mathbf{F}$, $v \in V$, and $av = 0$. Prove that $a = 0$ or $v = 0$.
- 3 Suppose $v, w \in V$. Explain why there exists a unique $x \in V$ such that $v + 3x = w$.
- 4 The empty set is not a vector space. The empty set fails to satisfy only one of the requirements listed in the definition of a vector space (1.20). Which one?
- 5 Show that in the definition of a vector space (1.20), the additive inverse condition can be replaced with the condition that

$$0v = 0 \text{ for all } v \in V.$$

Here the 0 on the left side is the number 0, and the 0 on the right side is the additive identity of V .

The phrase a “condition can be replaced” in a definition means that the collection of objects satisfying the definition is unchanged if the original condition is replaced with the new condition.

- 6 Let ∞ and $-\infty$ denote two distinct objects, neither of which is in $\mathbf{R}$. Define an addition and scalar multiplication on $\mathbf{R} \cup \{\infty, -\infty\}$ as you could guess from the notation. Specifically, the sum and product of two real numbers is as usual, and for $t \in \mathbf{R}$ define

$$t\infty = \begin{cases} -\infty & \text{if } t < 0, \\ 0 & \text{if } t = 0, \\ \infty & \text{if } t > 0, \end{cases} \quad t(-\infty) = \begin{cases} \infty & \text{if } t < 0, \\ 0 & \text{if } t = 0, \\ -\infty & \text{if } t > 0, \end{cases}$$

and

$$\begin{aligned} t + \infty &= \infty + t = \infty + \infty = \infty, \\ t + (-\infty) &= (-\infty) + t = (-\infty) + (-\infty) = -\infty, \\ \infty + (-\infty) &= (-\infty) + \infty = 0. \end{aligned}$$

With these operations of addition and scalar multiplication, is $\mathbf{R} \cup \{\infty, -\infty\}$ a vector space over $\mathbf{R}$? Explain.

- 7 Suppose S is a nonempty set. Let V^S denote the set of functions from S to V . Define a natural addition and scalar multiplication on V^S , and show that V^S is a vector space with these definitions.

- 8 Suppose V is a real vector space.

- The *complexification* of V , denoted by $V_{\mathbf{C}}$, equals $V \times V$. An element of $V_{\mathbf{C}}$ is an ordered pair (u, v) , where $u, v \in V$, but we write this as $u + iv$.
- Addition on $V_{\mathbf{C}}$ is defined by

$$(u_1 + iv_1) + (u_2 + iv_2) = (u_1 + u_2) + i(v_1 + v_2)$$

for all $u_1, v_1, u_2, v_2 \in V$.

- Complex scalar multiplication on $V_{\mathbf{C}}$ is defined by

$$(a + bi)(u + iv) = (au - bv) + i(av + bu)$$

for all $a, b \in \mathbf{R}$ and all $u, v \in V$.

Prove that with the definitions of addition and scalar multiplication as above, $V_{\mathbf{C}}$ is a complex vector space.

Think of V as a subset of $V_{\mathbf{C}}$ by identifying $u \in V$ with $u + i0$. The construction of $V_{\mathbf{C}}$ from V can then be thought of as generalizing the construction of $\mathbf{C}^n$ from $\mathbf{R}^n$.