

Def'n A permutation of $(1, \dots, n)$ is a list $(m_1, \dots, m_n)$ where each of $1, \dots, n$ appears exactly once. The set of all permutations of $(1, \dots, n)$ is denoted S_n .

Def'n The sign of a permutation $(m_1, \dots, m_n) \in S_n$ is the quantity $(-1)^t$ where t is the # of pairs (j, k) w/ $1 \leq j < k \leq n$ s.t. j appears after k in $(m_1, \dots, m_n)$.

Ex

- $(2, 1, 3, 4) \in S_4$ has sign -1 , since only $(2, 1)$ is out of order
- $(2, 3, \dots, n, 1) \in S_n$ has sign $(-1)^{n-1}$

Def'n If A is an $n \times n$ matrix. Then,

$$\text{"det"} A = \sum_{(m_1, \dots, m_n) \in S_n} \text{Sign}(m_1, \dots, m_n) A_{m_1, 1} \cdots A_{m_n, n}.$$

Ex

$$\begin{aligned} \text{"det"} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} &= \sum_{(m_1, m_2) \in S_2} \text{Sign}(m_1, m_2) A_{m_1, 1} A_{m_2, 2} \\ &= \text{Sign}(1, 2) A_{11} A_{22} + \text{Sign}(2, 1) A_{21} A_{12} \\ &= A_{11} A_{22} - A_{21} A_{12} \end{aligned}$$

Lemma $\S$ A is upper triangular. Then, "det" $A = A_{1,1} \cdots A_{n,n}$.

proof The permutation $(1, \dots, n) \in S_n$ contributes the term $\text{Sign}(1, \dots, n) A_{1,1} \cdots A_{n,n} = A_{1,1} \cdots A_{n,n}$. For any other permutation $(m_1, \dots, m_n) \in S_n$, there is some entry m_k s.t. $k < m_k$. But, $A_{m_k, k} = 0$ in this case since A is upper triangular. So, $\text{sign}(m_1, \dots, m_n) A_{m_1,1} \cdots A_{m_n,n} = 0$. Therefore,

$$\begin{aligned} \text{"det"} A &= \sum_{(m_1, \dots, m_n) \in S_n} \text{sign}(m_1, \dots, m_n) A_{m_1,1} \cdots A_{m_n,n} \\ &= A_{1,1} \cdots A_{n,n} \end{aligned}$$

Lemma "det" $(AB) = \text{"det"}(BA)$ for A, B $n \times n$ matrices.

Proof (sketch) Just write it out $\S$ use permutations to reorder everything.

Theorem "det" $(M(T, (u_1, \dots, u_n))) = \text{"det"}(M(T, (v_1, \dots, v_n)))$ for $T \in \mathcal{L}(V)$ $\S$ $(v_1, \dots, v_n), (u_1, \dots, u_n)$ two bases of V .

Proof By the change of basis formula, there is an invertible matrix C s.t. $A = C^{-1}BC$ where $A = M(T, (u_1, \dots, u_n))$ $\S$ $B = M(T, (v_1, \dots, v_n))$. So, using the last lemma, "det" $(A) = \text{"det"}(C^{-1}BC) = \text{"det"}(BC^{-1}C) = \text{"det"}(B)$.

Theorem "det" = det (i.e. $\text{det}(T) = \text{"det"}(M(T))$)

proof Use Jordan normal form along w/ our lemma re: upper triangular matrices!

Theorem $\& V$ is an inner product space $\& S \in \mathcal{L}(V)$ is unitary.
 Then, $|\det S| = 1$.

Proof $|\det S| = |\lambda_1 \cdots \lambda_n| = |\lambda_1| \cdots |\lambda_n| = 1 \cdots 1 = 1$

Where $\lambda_1, \dots, \lambda_n$ are the eigenvalues of S
 repeated according to their multiplicity.

Theorem $\& V$ is an inner product space $\& T \in \mathcal{L}(V)$.

Then, $\det T = s_1 \cdots s_n$ where $s_1, \dots, s_n$ are the singular values of T .

Proof By SVD, $\exists$ ONBs $e_1, \dots, e_n$ $\& f_1, \dots, f_n$ of V

s.t. $Tv = s_1 \langle v, e_1 \rangle f_1 + \dots + s_n \langle v, e_n \rangle f_n$. Take $S \in \mathcal{L}(V)$

so that $Se_n = f_n$ (this defines S since $e_1, \dots, e_n$ is a basis via
 the linear map lemma). Then,

$$\begin{aligned} \|Sv\|^2 &= \|S(\langle v, e_1 \rangle e_1 + \dots + \langle v, e_n \rangle e_n)\|^2 \\ &= \|\langle v, e_1 \rangle f_1 + \dots + \langle v, e_n \rangle f_n\|^2 \\ &= |\langle v, e_1 \rangle|^2 + \dots + |\langle v, e_n \rangle|^2 \\ &= \|v\|^2 \end{aligned}$$

So, S is unitary! Moreover, $T = S\sqrt{T^*T}$.

Hence, $\det T = \det(S\sqrt{T^*T}) = (\det S)(\det \sqrt{T^*T}) = \det(\sqrt{T^*T}) = s_1 \cdots s_n$.

Theorem $\& T \in \mathcal{L}(\mathbb{R}^n)$ $\& \Omega \subseteq \mathbb{R}^n$. Then,

Volume $T\Omega = |\det T|$ volume Ω .

Picture



lengths
 are singular values

so, $|\det T|$
 measures change
 in volume!