8D Trace: A Connection Between Matrices and Operators

8.47 definition: *trace of a matrix*

Suppose A is a square matrix with entries in $\mathbf{F}$. The *trace* of A, denoted by $\operatorname{tr} A$, is defined to be the sum of the diagonal entries of A.

$\boxed{\text{Ex}}$ Let $A = \begin{bmatrix} 3 & -1 & 2 \\ 3 & 2 & -3 \\ 1 & 2 & 0 \end{bmatrix}$. Then, $\operatorname{tr} A = 3+2+0=5$.

Easy stuff!

8.49 *trace of AB equals trace of BA*

Suppose A is an m-by-n matrix and B is an n-by-m matrix. Then

$$\operatorname{tr}(AB) = \operatorname{tr}(BA).$$

$\boxed{\text{Proof}}$ Idea: If $A = \begin{bmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & \ddots & \vdots \\ A_{n1} & \cdots & A_{nn} \end{bmatrix}$ & $B = \begin{bmatrix} B_{11} & \cdots & B_{1n} \\ \vdots & & \vdots \\ B_{n1} & \cdots & B_{nn} \end{bmatrix}$

Then $\operatorname{tr} A = \operatorname{tr} \hat{A}$ & $\operatorname{tr} B = \operatorname{tr} \check{B}$ w/ $\hat{A} = \begin{bmatrix} A_{11} & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & A_{nn} \end{bmatrix}$ $\check{B} = \begin{bmatrix} B_{11} & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & B_{nn} \end{bmatrix}$

Since $\hat{A}$ & $\check{B}$ commute, we are done!

More precisely: $(AB)_{jj} = \sum_{k=1}^{n} A_{jk} B_{kj}$, so

$$\operatorname{tr} AB = \sum_{j=1}^{m} \sum_{k=1}^{n} A_{jk} B_{kj} = \sum_{k=1}^{n} \sum_{j=1}^{m} B_{kj} A_{jk} = \operatorname{tr} BA.$$

8.50 *trace of matrix of operator does not depend on basis*

Suppose $T \in \mathcal{L}(V)$. Suppose $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$ are bases of V. Then

$$\operatorname{tr} \mathcal{M}(T, (u_1, \ldots, u_n)) = \operatorname{tr} \mathcal{M}(T, (v_1, \ldots, v_n)). \checkmark$$

$\boxed{\text{Proof}}$ Let $A = \mathcal{M}(T, (u_1, \ldots, u_n))$ & $B = \mathcal{M}(T, (v_1, \ldots, v_n))$.

By the change of basis formula (back in ch3), there is an invertible matrix C such that $A = C^{-1}BC$. Thus, using the previous result,

$\operatorname{tr}(A) = \operatorname{tr}(C^{-1}BC) = \operatorname{tr}(C^{-1}CB) = \operatorname{tr}(B)$.

8.51 definition: *trace of an operator*

Suppose $T \in \mathcal{L}(V)$. The *trace* of T, denote $\operatorname{tr} T$, is defined by

$$\operatorname{tr} T = \operatorname{tr} \mathcal{M}(T, (v_1, \dots, v_n)),$$

where $v_1, \dots, v_n$ is any basis of V.

8.52 *on complex vector spaces, trace equals sum of eigenvalues*

Suppose $\mathbf{F} = \mathbf{C}$ and $T \in \mathcal{L}(V)$. Then $\operatorname{tr} T$ equals the sum of the eigenvalues of T, with each eigenvalue included as many times as its multiplicity.

$\leftarrow$ This says that if $\lambda_1, \dots, \lambda_m$ are the distinct e-vals of T, then $\operatorname{tr} T = \sum_{i=1}^{m} \lambda_i \dim\left((T - \lambda_i I)^{\dim V}\right) = \sum_{i=1}^{m} \lambda_i \dim G(\lambda_i, T)$

$\boxed{\text{Proof}}$ Using the Jordan form, there is a basis of V for which the matrix of T is upper triangular & has on its diagonal its eigenvalues repeated as many times as their multiplicities. So, since the trace does not depend on the choice of basis, using this basis delivers the result!

Jordan Form | $\mathcal{M}(T) = \begin{bmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_m \end{bmatrix}$ w/ $A_i = \begin{bmatrix} \lambda_i & & * \\ & \ddots & \\ & & \lambda_i \end{bmatrix} \Big\} \dim G(\lambda_i, T)$

8.54 *trace and characteristic polynomial*

Suppose $\mathbf{F} = \mathbf{C}$ and $T \in \mathcal{L}(V)$. Let $n = \dim V$. Then $\operatorname{tr} T$ equals the negative of the coefficient of z^{n-1} in the characteristic polynomial of T.

$\boxed{\text{Proof}}$ $\mathcal{S}$ $\mathbb{F} = \mathbb{C}$, $T \in \mathcal{L}(V)$, $n = \dim V$, & $\lambda_1, \dots, \lambda_m$ are the eigenvalues of T. Then, the characteristic poly. of T is $(z - \lambda_1)^{\dim G(\lambda_1, T)} \cdots (z - \lambda_m)^{\dim G(\lambda_m, T)}$.

Expanding, we get a poly of the form

$$z^n - \underbrace{(\underbrace{\lambda_1 + \dots + \lambda_1}_{\dim G(\lambda_1, T)} + \dots + \underbrace{\lambda_m + \dots + \lambda_m}_{\dim G(\lambda_m, T)})}_{\operatorname{tr} T} z^{n-1} + \dots + \left((\underbrace{\lambda_1 \cdots \lambda_1}_{\dim G(\lambda_1, T)}) \cdots (\underbrace{\lambda_m \cdots \lambda_m}_{\dim G(\lambda_m, T)})\right)(-1)^n$$

not really showing this bit

Or, if we let $\lambda_1, \dots, \lambda_n$ be the eigenvalues of T w/ repetition according to multiplicity, we get the slightly cleaner form:

$$z^n - (\lambda_1 + \dots + \lambda_n) z^{n-1} + \dots + (-1)^n (\lambda_1 \cdots \lambda_n)$$

9C Determinants

Defining the Determinant

Def'n Suppose $T \in \mathcal{L}(V)$. If $\mathbb{F} = \mathbb{C}$, then the determinant of T, denoted $\det T$, is the product of the eigenvalues of T, each repeated according to their multiplicity. If $\mathbb{F} = \mathbb{R}$, then $\det(T) = \det(T_{\mathbb{C}})$.

Theorem The determinant of $T \in \mathcal{L}(V)$ is, up to a sign, the constant term of the characteristic poly of T.

Proof (Look at the last one)

Theorem An operator is invertible if & only if its determinant is nonzero.

Proof By definition, $\det T = 0$ for $T \in \mathcal{L}(V)$ if & only if 0 is an eigenvalue of T (since a product of complex #'s is zero if & only if one is).

Hence, $\det T \neq 0 \iff 0$ not an e-val of T
$\iff T$ is invertible.

Theorem Suppose $T \in \mathcal{L}(V)$. Then, the characteristic poly. of T is $\det(zI - T)$.

Proof Suppose V is a $\mathbb{C}$-V.S. & $T \in \mathcal{L}(V)$. ⟵ If $\mathbb{F} = \mathbb{R}$, then the same argument holds w/ T replaced by $T_{\mathbb{C}}$

λ is an e-val of $T \iff z - \lambda$ is an e-val of $zI - T$ (since $-(T - \lambda I) = (zI - T) - (z - \lambda)I$). So,

$\det(zI - T) = (z - \lambda_1) \cdots (z - \lambda_n)$ where the λ_i's are e-vals of T repeated according to their multiplicity. But this is just the characteristic poly of T!

Theorem Let $T \in \mathcal{L}(V)$ & define $g: \mathbb{F} \to \mathbb{F}$ by
$$g(x) = \det(I + xT).$$
Then, $g'(0) = \operatorname{tr} T$.

proof Notice that $I + xT$ has eigenvalues $1+x\lambda_1, \dots, 1+x\lambda_n$ (each repeated according to their multiplicity)

So, $g(x) = (1+x\lambda_1) \cdots (1+x\lambda_n)$ and

$$\begin{aligned}
g'(0) &= \lim_{x \to 0} \frac{g(x) - g(0)}{x - 0} \\
&= \lim_{x \to 0} \frac{(1+x\lambda_1)\cdots(1+x\lambda_n) - 1}{x} \\
&= \lim_{x \to 0} \frac{\cancel{1} + (x\lambda_1 + \cdots + x\lambda_n) + \overbrace{\cdots + (x\lambda_1)\cdots(x\lambda_n)}^{\text{has } x \text{ terms of deg 2 or more}} - \cancel{1}}{x} \\
&= \lim_{x \to 0} (\lambda_1 + \cdots + \lambda_n) + (\text{stuff w/ } x\text{'s}) \\
&= \lambda_1 + \cdots \lambda_n \\
&= \operatorname{tr} T.
\end{aligned}$$

So, the trace is the rate of change of the determinant as it starts moving from the identity in the direction of your operator

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Fact: There is a fancy version of this known as Jacobi's formula, which says that if $A : \mathbb{R} \to \mathcal{L}(V)$, then $\frac{d}{dt} \det A(t) = \operatorname{tr}\left(\operatorname{adj}(A(t)) \frac{dA}{dt}(t)\right)$ where $\operatorname{adj} A$ is the adjugate of A.