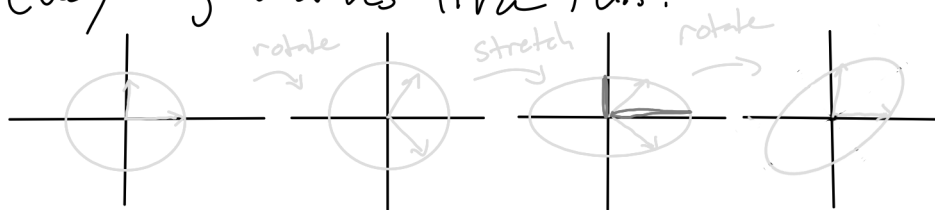


7E Singular Value Decomposition

Idea For linear maps between $\mathbb{R}^n$ & $\mathbb{R}^m$, everything works like this:



Today, we will show this is true & that an analog of it holds for maps in $\mathcal{L}(V, W)$ for V, W inner product spaces of finite dim.

7.64 properties of T^*T

Suppose $T \in \mathcal{L}(V, W)$. Then

- (a) T^*T is a positive operator on V ;
- (b) $\text{null } T^*T = \text{null } T$;
- (c) $\text{range } T^*T = \text{range } T^*$;
- (d) $\dim \text{range } T = \dim \text{range } T^* = \dim \text{range } T^*T$.

Proof (a) Notice $(T^*T)^* = T^*(T^*)^* = T^*T$, so T^*T is self-adjoint. If $v \in V$, $\langle T^*Tv, v \rangle = \langle Tv, Tv \rangle = \|Tv\|^2 \geq 0$. Hence, T^*T is positive.

(b) If $v \in \text{null } T$, $Tv = 0$ so $T^*Tv = T^*0 = 0$ & $v \in \text{null } T^*T$. Hence, $\text{null } T \subseteq \text{null } T^*T$. Now, if $v \in \text{null } T^*T$. Then, we have $0 = \langle 0, v \rangle = \langle T^*Tv, v \rangle = \langle Tv, Tv \rangle = \|Tv\|^2$. So, $Tv = 0$ & $v \in \text{null } T$. Hence $\text{null } T^*T \subseteq \text{null } T$. So, $\text{null } T = \text{null } T^*T$.

(c) Since T^*T is self-adjoint (by part (a)), we have $\text{range } T^*T = (\text{null } T^*T)^\perp = (\text{null } T)^\perp = \text{range } T^*$.

(d) By part (c), $\dim \text{range } T^* = \dim \text{range } T^*T$. Now, observe $\dim \text{range } T = \dim (\text{null } T^*)^\perp = \dim W - \dim \text{null } T^* = \dim \text{range } T^*$.

7.65 definition: singular values

Suppose $T \in \mathcal{L}(V, W)$. The singular values of T are the nonnegative square roots of the eigenvalues of T^*T , listed in decreasing order, each included as many times as the dimension of the corresponding eigenspace of T^*T .

i.e. if $\lambda_1, \dots, \lambda_k$ are the eigenvalues of T^*T , and $\dim E(\lambda_i, T^*T) = n_i$, then $\underbrace{\sqrt{\lambda_1}, \dots, \sqrt{\lambda_1}}_{n_1}, \dots, \underbrace{\sqrt{\lambda_k}, \dots, \sqrt{\lambda_k}}_{n_k}$ are the singular values of T , assuming that $\lambda_1 \geq \dots \geq \lambda_k$.

7.68 role of positive singular values

Suppose that $T \in \mathcal{L}(V, W)$. Then

- (a) T is injective $\iff 0$ is not a singular value of T ;
- (b) the number of positive singular values of T equals $\dim \text{range } T$;
- (c) T is surjective $\iff$ number of positive singular values of T equals $\dim W$.

Proof (a) A linear map $T \in \mathcal{L}(V, W)$ is injective iff $\text{null } T = \{0\}$.

So, T is injective $\iff \text{null } T = \{0\} \iff \text{null } T^*T = \{0\} \iff 0$ not an eigenvalue of T^*T
 $\iff 0$ not a singular value of T

(b) Using the spectral theorem on T^*T , there is a basis of e -vectors of T^*T s.t. $M(T)$ is diagonal w/ its eigenvalues on the diagonal. Hence, $\dim \text{range } T^*T = \dim \text{span}(\text{columns of } M(T))$
 $= \# \text{ of columns} - \# \text{ of times } 0 \text{ appears on the diagonal of } M(T)$
 $= \# \text{ of positive } e\text{-vals on diagonal of } M(T)$
 $= \# \text{ of positive singular values}$

Now, just use $\dim \text{range } T = \dim \text{range } T^*T$

(c) Obvious from (b).

list of eigenvalues	list of singular values
context: <u>vector spaces</u>	context: <u>inner product spaces</u>
defined only for <u>linear maps from a vector space to itself</u> (i.e. operators)	defined for <u>linear maps from an inner product space to a possibly different inner product space</u>
can be <u>arbitrary real numbers</u> (if $F = \mathbb{R}$) or <u>complex numbers</u> (if $F = \mathbb{C}$)	are <u>nonnegative numbers</u>
<u>can be the empty list</u> if $F = \mathbb{R}$	<u>length of list equals dimension of domain</u>
<u>includes 0</u> $\iff$ <u>operator is not invertible</u>	<u>includes 0</u> $\iff$ <u>linear map is not injective</u>
<u>no standard order</u> , especially if $F = \mathbb{C}$	<u>always listed in decreasing order</u>

bookkeeping, not really a "fact" in any substantial sense.

7.69 isometries characterized by having all singular values equal 1

Suppose that $S \in \mathcal{L}(V, W)$. Then

S is an isometry $\iff$ all singular values of S equal 1.

proof

S is an isometry $\iff S^*S = I$
 $\iff S^*S$ has all eigenvalues equal to 1
 $\iff S$ has all singular values equal to 1

(the first equivalence is something we proved last time).

7.70 singular value decomposition

Suppose $T \in \mathcal{L}(V, W)$ and the positive singular values of T are $s_1, \dots, s_m$. Then there exist orthonormal lists $e_1, \dots, e_m$ in V and $f_1, \dots, f_m$ in W such that

$$7.71 \quad Tv = s_1 \langle v, e_1 \rangle f_1 + \dots + s_m \langle v, e_m \rangle f_m$$

for every $v \in V$.

proof

Let $s_1, \dots, s_n$ be the singular values of T (hence $n = \dim V$). By the spectral theorem applied to the positive operator T^*T , there is an ONB $e_1, \dots, e_n$ of V s.t. $T^*Te_k = s_k^2 e_k$ (since s_k is the square root of an eigenvalue of T^*T). Now, for $k \in \{1, \dots, m\}$ (where $s_1, \dots, s_m$ are positive & $s_{m+1}, \dots, s_n$ are all 0), set $f_k = \frac{Te_k}{s_k}$.

Then, if $j, k \in \{1, \dots, m\}$, we have

$$\langle f_j, f_k \rangle = \langle \frac{1}{s_j} Te_j, \frac{1}{s_k} Te_k \rangle = \frac{1}{s_j s_k} \langle e_j, T^*Te_k \rangle = \frac{s_k^2}{s_j s_k} \langle e_j, e_k \rangle = \begin{cases} 1 & j=k \\ 0 & j \neq k \end{cases}$$

So, $f_1, \dots, f_m$ is an orthonormal list in W .

Let $v \in V$. Since $e_1, \dots, e_n$ is an ONB, we get

$$v = \langle v, e_1 \rangle e_1 + \dots + \langle v, e_n \rangle e_n$$

$$Tv = \langle v, e_1 \rangle Te_1 + \dots + \langle v, e_n \rangle Te_n$$

$$Tv = s_1 \langle v, e_1 \rangle f_1 + \dots + s_m \langle v, e_m \rangle f_m$$

Since $s_k f_k = Te_k$
 & this works if $s_k = 0$ as
 well since $Te_k = 0$
 b/c $\text{null } T = \text{null } T^*T$

<u>spectral theorem</u>	<u>singular value decomposition</u>
describes <u>only self-adjoint operators</u> (when $F = \mathbf{R}$) or <u>normal operators</u> (when $F = \mathbf{C}$)	describes <u>arbitrary linear maps</u> from an <u>inner product space</u> to a <u>possibly different inner product space</u>
<u>produces a single orthonormal basis</u>	produces <u>two orthonormal lists</u> , one for <u>domain space</u> and one for <u>range space</u> , that are <u>not necessarily the same even when range space equals domain space</u>
<u>different proofs depending on whether $F = \mathbf{R}$ or $F = \mathbf{C}$</u>	<u>same proof works regardless of whether $F = \mathbf{R}$ or $F = \mathbf{C}$</u>

7.75 *singular value decomposition of adjoint and pseudoinverse*

Suppose $T \in \mathcal{L}(V, W)$ and the positive singular values of T are $s_1, \dots, s_m$. Suppose $\underline{e_1, \dots, e_m}$ and $\underline{f_1, \dots, f_m}$ are orthonormal lists in V and W such that

$$7.76 \quad \underline{Tv = s_1 \langle v, e_1 \rangle f_1 + \dots + s_m \langle v, e_m \rangle f_m}$$

for every $v \in V$. Then

$$7.77 \quad \underline{T^*w = s_1 \langle w, f_1 \rangle e_1 + \dots + s_m \langle w, f_m \rangle e_m}$$

and

$$7.78 \quad T^\dagger w = \frac{\langle w, f_1 \rangle}{s_1} e_1 + \dots + \frac{\langle w, f_m \rangle}{s_m} e_m$$

for every $w \in W$.

7.80 *matrix version of SVD*

Suppose A is an M -by- n matrix of rank $m \geq 1$. Then there exist an M -by- m matrix B with orthonormal columns, an m -by- m diagonal matrix D with positive numbers on the diagonal, and an n -by- m matrix C with orthonormal columns such that

$$A = BDC^*.$$

This is the picture from the start of class!