

7D Isometries, Unitary Operators, and Matrix Factorization

Isometries

7.44 definition: isometry

A linear map $S \in \mathcal{L}(V, W)$ is called an *isometry* if

$$\|Sv\|_W = \|v\|_V$$

for every $v \in V$. In other words, a linear map is an isometry if it preserves norms.

Note $\S S v = 0$. Then, $\|v\| = \|Sv\| = 0$, so $v = 0$.
That is, $\text{null } S = \{0\}$ & S is injective!

7.49 characterization of isometries

Suppose $S \in \mathcal{L}(V, W)$. Suppose $e_1, \dots, e_n$ is an orthonormal basis of V and $f_1, \dots, f_m$ is an orthonormal basis of W . Then the following are equivalent.

- (a) S is an isometry.
- (b) $S^*S = I$.
- (c) $\langle Su, Sv \rangle = \langle u, v \rangle$ for all $u, v \in V$.
- (d) $Se_1, \dots, Se_n$ is an orthonormal list in W .
- (e) The columns of $\mathcal{M}(S, (e_1, \dots, e_n), (f_1, \dots, f_m))$ form an orthonormal list in $\mathbb{F}^m$ with respect to the Euclidean inner product.

Proof $(a) \Rightarrow (b)$ If $v \in V$, then $\langle (I - S^*S)v, v \rangle = \langle v, v \rangle - \langle S^*Sv, v \rangle$
 $= \langle v, v \rangle - \langle Sv, Sv \rangle$
 $= \|v\|^2 - \|Sv\|^2$
 $= 0$
 Since $I - S^*S$ is self-adjoint & satisfies $\langle (I - S^*S)v, v \rangle = 0$ for $\forall v \in V$.

So, $I - S^*S$ is the zero operator, that is $S^*S = I$.

$(b) \Rightarrow (c)$ $\S S^*S = I$. $\langle Su, Sv \rangle = \langle S^*Su, v \rangle = \langle u, v \rangle$

$(c) \Rightarrow (d)$ If $e_1, \dots, e_n$ is an ONB of V & $\langle Su, Sv \rangle = \langle u, v \rangle \forall u, v \in V$,
 then $\langle Se_i, Se_j \rangle = \langle e_i, e_j \rangle = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}$. So, $Se_1, \dots, Se_n$ is an orthonormal list in W .

$(d) \Rightarrow (e)$ Let $A = \mathcal{M}(S)$. Then, for $k, r \in \{1, \dots, n\}$, we have

$$A_{\cdot, k} \cdot A_{\cdot, r} = \sum_{j=1}^n A_{j, k} \overline{A_{j, r}} = \left\langle \sum_{j=1}^n A_{j, k} f_j, \sum_{i=1}^n A_{i, r} f_i \right\rangle = \langle Se_k, Se_r \rangle = \begin{cases} 1 & k=r \\ 0 & k \neq r \end{cases}$$

$$\sum_{j=1}^n A_{j, k} \left\langle f_j, \sum_{i=1}^n A_{i, r} f_i \right\rangle = \sum_{j=1}^n A_{j, k} \sum_{i=1}^n \overline{A_{i, r}} \langle f_j, f_i \rangle$$

Proof (continued)

(c) $\Rightarrow$ (a) If the columns of $M(S)$ are an orthonormal list in $\mathbb{F}^m$, then $Se_1, \dots, Se_n$ is an orthonormal list in W .

So, for $v \in V$, $v = \langle v, e_1 \rangle e_1 + \dots + \langle v, e_n \rangle e_n$ &

$sv = \langle v, e_1 \rangle Se_1 + \dots + \langle v, e_n \rangle Se_n$. So, $\|v\|^2 = |\langle v, e_1 \rangle|^2 + \dots + |\langle v, e_n \rangle|^2$

& $\|sv\|^2 = |\langle v, e_1 \rangle|^2 + \dots + |\langle v, e_n \rangle|^2$ & S is an isometry.

Unitary Operators

7.51 definition: unitary operator

An operator $S \in \mathcal{L}(V)$ is called unitary if S is an invertible isometry.

7.53 characterization of unitary operators

Suppose $S \in \mathcal{L}(V)$. Suppose $e_1, \dots, e_n$ is an orthonormal basis of V . Then the following are equivalent.

- (a) S is a unitary operator.
- (b) $S^*S = SS^* = I$.
- (c) S is invertible and $S^{-1} = S^*$.
- (d) $Se_1, \dots, Se_n$ is an orthonormal basis of V .
- (e) The rows of $M(S, (e_1, \dots, e_n))$ form an orthonormal basis of $\mathbb{F}^n$ with respect to the Euclidean inner product.
- (f) S^* is a unitary operator.

Proof $(a) \Rightarrow (b)$ If S is unitary, it is an isometry. So, $S^*S = I$. Since unitary operators are invertible, this gives $S^* = S^*SS^{-1} = S^{-1}$. So, $S^*S = SS^* = I$.

$(b) \Rightarrow (c)$ $\S$ $S^*S = SS^* = I$. So, by definition, $S^{-1} = S^*$.

$(c) \Rightarrow (d)$ Since $S^{-1} = S^*$ implies $S^*S = I$, the previous result delivers this via $(b) \Leftrightarrow (d)$ in the last result.

$(d) \Rightarrow (e)$ This is a direct consequence of the last result (particularly, $(d) \Rightarrow (e)$ there).

$(e) \Rightarrow (f)$ $M(S)$ has ~~rows~~ ^{rows} an ONB implies $M(S^*) = \overline{M(S)}^t$ and an ONB. So, S^* is an isometry $\S$ given both the rows $\S$ columns of $M(S)$ are ONB $\Rightarrow S^*$ is invertible. (doing this again w/ S^* replacing S)

$(f) \Rightarrow (a)$

Just do $(a) \Rightarrow (f)$ to S^* to get $(S^*)^* = S$ is unitary

7.54 eigenvalues of unitary operators have absolute value 1

Suppose λ is an eigenvalue of a unitary operator. Then $|\lambda| = 1$.

Proof $\S$ S is unitary $\S$ $Sv = \lambda v$ for $v \neq 0$ $\&$ $\lambda \in \mathbb{F}$. Then,

$$1 = \frac{\|Sv\|}{\|v\|} = \frac{\|\lambda v\|}{\|v\|} = \frac{|\lambda| \|v\|}{\|v\|} = |\lambda|, \text{ since } \|Sv\| = \|v\|.$$

7.55 description of unitary operators on complex inner product spaces

Suppose $\mathbf{F} = \mathbf{C}$ and $S \in \mathcal{L}(V)$. Then the following are equivalent.

- (a) S is a unitary operator.
- (b) There is an orthonormal basis of V consisting of eigenvectors of S whose corresponding eigenvalues all have absolute value 1.

Proof $\S$ S is unitary. Then, $S^*S = I = SS^*$, so S is normal. Thus, by the complex spectral theorem, $\exists$ an ONB of e-vec of S . Since all e-values of S have modulus 1, (a) $\Rightarrow$ (b).

Now, $\S$ $\exists$ an ONB of e-vectors of S w/ correspondingly e-val's all modulus 1. Then, $\lambda_1 e_1, \dots, \lambda_n e_n$ is also an ONB, since $\langle \lambda_i e_i, \lambda_k e_k \rangle = \langle e_i, e_k \rangle \lambda_i \overline{\lambda_k} = \begin{cases} 0 & i \neq k \\ |\lambda_i|^2 & i = k \end{cases} \underset{1}{=} 1$

So, since $\lambda_1 e_1, \dots, \lambda_n e_n$ is just $S e_1, \dots, S e_n$, S is unitary by the previous result.

QR Factorization

7.56 definition: unitary matrix

An n -by- n matrix is called unitary if its columns form an orthonormal list in $\mathbb{F}^n$.

← i.e. if it is the matrix of a unitary operator w.r.t. an ONB.

7.57 characterizations of unitary matrices

Suppose Q is an n -by- n matrix. Then the following are equivalent.

- (a) Q is a unitary matrix.
- (b) The rows of Q form an orthonormal list in $\mathbb{F}^n$.
- (c) $\|Qv\| = \|v\|$ for every $v \in \mathbb{F}^n$.
- (d) $Q^*Q = QQ^* = I$, the n -by- n matrix with 1's on the diagonal and 0's elsewhere.

7.58 QR factorization

Suppose A is a square matrix with linearly independent columns. Then there exist unique matrices Q and R such that Q is unitary, R is upper triangular with only positive numbers on its diagonal, and

$$A = QR.$$

True, but we want prove it

proof Let $A \in \mathbb{F}^{n \times n}$ have linearly indep. cols, $v_1, \dots, v_n$.

Use GS to get $e_1, \dots, e_n$ an ONB of $\mathbb{F}^n$ s.t.

$\text{span}(v_1, \dots, v_k) = \text{span}(e_1, \dots, e_k)$. Let R be the matrix w/ $R_{jk} = \langle v_k, e_j \rangle$. Then, R is UT, since for $k < j$,

e_j is orthogonal to everything in $\text{span}(v_1, \dots, v_k) = \text{span}(e_1, \dots, e_k)$.

Then, if Q is the matrix w/ columns $e_1, \dots, e_n$, it is unitary (since its cols are an ONB) & the k th col

of QR is just $\underbrace{\langle v_k, e_1 \rangle}_{\text{}} e_1 + \dots + \underbrace{\langle v_k, e_k \rangle}_{\text{}} e_k = v_k$. So, $QR = A$.

Ex Let $A = \begin{bmatrix} 12 & 1 \\ 0 & 1 & -4 \\ 0 & 3 & 2 \end{bmatrix}$. Applying GS to

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, v_3 = \begin{bmatrix} 1 \\ -4 \\ 2 \end{bmatrix}, \text{ we get}$$

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 1 \\ 1/\sqrt{10} \\ -3/\sqrt{10} \end{bmatrix}, e_3 = \begin{bmatrix} 0 \\ 3/\sqrt{10} \\ 1/\sqrt{10} \end{bmatrix}. \text{ Then,}$$

$$Q = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1/\sqrt{10} & 3/\sqrt{10} \\ 0 & -3/\sqrt{10} & 1/\sqrt{10} \end{bmatrix} \text{ \& } R = \begin{bmatrix} \langle v_1, e_1 \rangle & \langle v_2, e_1 \rangle & \langle v_3, e_1 \rangle \\ \langle v_1, e_2 \rangle & \langle v_2, e_2 \rangle & \langle v_3, e_2 \rangle \\ \langle v_1, e_3 \rangle & \langle v_2, e_3 \rangle & \langle v_3, e_3 \rangle \end{bmatrix} = \begin{bmatrix} 12 & 1 & 1 \\ 0 & \sqrt{10} & \frac{\sqrt{10}}{5} \\ 0 & 0 & \frac{7\sqrt{10}}{5} \end{bmatrix}$$

You can check that $A = QR$.

This is nice! If we want to solve $Ax=b$,
then $QRx=b$ \& $Rx=Q^*b$ since Q is unitary.

As $Rx=y$ is easy to solve (R is UT), we
can do this quickly!

$$Rx = \begin{bmatrix} 12 & 1 & 1 \\ 0 & \sqrt{10} & \frac{\sqrt{10}}{5} \\ 0 & 0 & \frac{7\sqrt{10}}{5} \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a+2b+c \\ \sqrt{10}b + \frac{\sqrt{10}}{5}c \\ \frac{7\sqrt{10}}{5}c \end{bmatrix}$$

So, we can solve $Rx = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$ via $C = \frac{5}{7\sqrt{10}} z_3$
 $D = \frac{z_2 - \left(\frac{\sqrt{10}}{5}\right)\left(\frac{5}{7\sqrt{10}}\right)z_3}{\sqrt{10}}$
 $a = (\text{same process})$