

7C Positive Operators

7.34 definition: positive operator

An operator $T \in \mathcal{L}(V)$ is called positive if T is self-adjoint and

$$\underbrace{\langle Tv, v \rangle}_{\in \mathbb{R}} \geq 0$$

for all $v \in V$.

Note If V is an inner product space over $\mathbb{C}$, then we can drop the requirement that T is self-adjoint since $\langle Tv, v \rangle \in \mathbb{R}$ for all $v \in V$ implies this in that case. Be careful to not make a mistake and forget this! $\mathbb{C}$ doesn't have " $\leq$ " like $\mathbb{R}$ does!

7.36 definition: square root

An operator $\underline{R}$ is called a square root of an operator $\underline{T}$ if $\underline{R}^2 = \underline{T}$.

Ex Let $T \in \mathcal{L}(\mathbb{F}^3)$ be $T(x_1, x_2, x_3) = (x_3, 0, 0)$.
Let $R \in \mathcal{L}(\mathbb{F}^3)$ be $R(x_1, x_2, x_3) = (x_2, x_3, 0)$. Then,
 $R^2(x_1, x_2, x_3) = R(x_2, x_3, 0) = (x_3, 0, 0) = T(x_1, x_2, x_3)$.
So, $R^2 = T$ & R is a square root of T .

Notice: If $R'(x_1, x_2, x_3) = (-x_2, -x_3, 0)$, then
 $(R')^2(x_1, x_2, x_3) = R'(-x_2, -x_3, 0) = (x_3, 0, 0) = T(x_1, x_2, x_3)$,
so R' is a square root of T as well.

Let $T \in \mathcal{L}(V)$. Then the following are equivalent.

- (a) T is a positive operator.
- (b) T is self-adjoint and all eigenvalues of T are nonnegative.
- (c) With respect to some orthonormal basis of V , the matrix of T is a diagonal matrix with only nonnegative numbers on the diagonal.
- (d) T has a positive square root.
- (e) T has a self-adjoint square root.
- (f) $T = R^*R$ for some $R \in \mathcal{L}(V)$.

Proof $(a) \Rightarrow (b)$ That a positive operator is self adjoint is immediate from the definition. $\S$ T is positive $\&$ λ is an e-val of T w/ e-vec v . Then $0 \leq \langle Tv, v \rangle = \langle \lambda v, v \rangle = \lambda \|v\|^2$

So, $\lambda \geq 0$.

$(b) \Rightarrow (c)$ Spectral Theorem

$(c) \Rightarrow (d)$ By the linear map lemma, there is a $R \in \mathcal{L}(V)$ st.

$Re_k = \sqrt{\lambda_k} e_k$ where $e_1, \dots, e_n$ is the assumed ONB of V st. $M(T)$ is diagonal w/ λ_k its k^{th} diagonal entry. Certainly,

R is positive: $\langle Re_k, e_k \rangle = \sqrt{\lambda_k} \langle e_k, e_k \rangle = \sqrt{\lambda_k} \|e_k\|^2 = \sqrt{\lambda_k} \geq 0$.

Moreover, $R^2 e_k = \sqrt{\lambda_k}^2 e_k = \lambda_k e_k = Te_k$, so $T = R^2$.

$(d) \Rightarrow (e)$ Positive operators are self adjoint.

$(e) \Rightarrow (f)$ $\S$ $T = R^2$ $\&$ R is self-adjoint, then,

$R = R^*$, so $R^2 = R^*R = T$.

$(f) \Rightarrow (a)$ $\S$ $T = R^*R$. Then, $T^* = R^*(R^*)^* = R^*R = T$.

So, T is self-adjoint. Moreover, $\langle Tv, v \rangle = \langle R^*Rv, v \rangle = \langle Rv, Rv \rangle = \|Rv\|^2 \geq 0$.

So, T is positive.

Every positive operator on V has a unique positive square root.

Proof $\S$ $T \in \mathcal{L}(V)$ is a positive operator. Let R be a positive square root of T , which we know exists by the last result. We show uniqueness. Let λ be an eigenvalue of T w/ eigenvector v . We will show that $Rv = \sqrt{\lambda} v$ which shows R is unique, since V has a basis of eigenvectors of T (by the spectral theorem).

That $Rv = \sqrt{\lambda} v$ is true follows from applying the spectral theorem to R to get an ONB of e-vecs of R for V .

Let $e_1, \dots, e_n$ be this basis. Since R is positive, $\exists$ nonnegative $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ s.t. $Re_k = \sqrt{\lambda_k} e_k$. So, write $v = a_1 e_1 + \dots + a_n e_n$, and notice $R^2 v = a_1 \lambda_1 e_1 + \dots + a_n \lambda_n e_n$.

$\S$ $Tv = \lambda v$ implies $\lambda v = a_1 \lambda e_1 + \dots + a_n \lambda e_n = Tv = R^2 v = a_1 \lambda_1 e_1 + \dots + a_n \lambda_n e_n$.

So, $a_1(\lambda - \lambda_1)e_1 + \dots + a_n(\lambda - \lambda_n)e_n = 0$ $\S$ since $e_1, \dots, e_n$ is a basis, if $a_i \neq 0$, then $\lambda = \lambda_i$. So, $Rv = a_1 \sqrt{\lambda_1} e_1 + \dots + a_n \sqrt{\lambda_n} e_n = \sqrt{\lambda} v$

(since, if $a_i = 0$, we can still replace $\sqrt{\lambda_k}$ w/ $\sqrt{\lambda}$ here $\nearrow$ w/o changing anything).

7.40 notation: $\sqrt{T}$

For T a positive operator, $\sqrt{T}$ denotes the unique positive square root of T .

7.43 T positive and $\langle Tv, v \rangle = 0 \implies Tv = 0$

Suppose T is a positive operator on V and $v \in V$ is such that $\langle Tv, v \rangle = 0$. Then $Tv = 0$.

Proof

$$0 = \langle Tv, v \rangle = \langle \sqrt{T} \sqrt{T} v, v \rangle = \langle \sqrt{T} v, \sqrt{T} v \rangle = \|\sqrt{T} v\|^2$$

$$\text{So, } \sqrt{T} v = 0. \text{ Thus, } Tv = \sqrt{T} \sqrt{T} v = \sqrt{T} 0 = 0.$$

Ex

Let $T \in \mathcal{L}(\mathbb{R}^2)$ be $T(x, y) = (x+y, x+y)$.

$$\langle T(x, y), (x, y) \rangle = \langle (x+y, x+y), (x, y) \rangle = x(x+y) + y(x+y) = (x+y)^2 \geq 0$$

So, T is positive, since wrt. the standard basis $(1, 0), (0, 1)$ of $\mathbb{R}^2$, $M(T) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = M(T)^t = M(T^*)$. Now, $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}), (\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$ is a basis of $\mathbb{R}^2$ that consists only of e-vectors of T w/ corresponding e-val's $2, 0$. Wrt. this basis, T has a matrix $M(T) = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$, so $M(\sqrt{T}) = \begin{bmatrix} \sqrt{2} & 0 \\ 0 & 0 \end{bmatrix}$ wrt. this basis.

$$\text{Wrt. the standard basis of } \mathbb{R}^2, M(\sqrt{T}) = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}.$$