7B Spectral Theorem

Real Spectral Theorem

7.26 *invertible quadratic expressions*

Suppose $T \in \mathcal{L}(V)$ is self-adjoint and $b, c \in \mathbf{R}$ are such that $b^2 < 4c$. Then

$$T^2 + bT + cI$$

is an invertible operator.

$\boxed{\text{proof}}$ $\oint v \neq 0$. Then,

$$\begin{aligned}
\langle (T^2 + bT + cI)v, v \rangle &= \langle T^2 v, v \rangle + b\langle Tv, v\rangle + c\langle v, v\rangle \\
&= \langle Tv, T^* v\rangle + b\langle Tv, v\rangle + c\langle v, v\rangle \\
&= \langle Tv, Tv\rangle + b\langle Tv, v\rangle + c\langle v, v\rangle \\
&= \|Tv\|^2 + b\langle Tv, v\rangle + c\|v\|^2 \\
&\geq \|Tv\|^2 - |b||\langle Tv, v\rangle| + c\|v\|^2 \\
&\geq \|Tv\|^2 - |b|\,\|Tv\|\,\|v\| + c\|v\|^2 \\
&= \left(\|Tv\| - \frac{|b|\,\|v\|}{2}\right)^2 + \left(c - \frac{b^2}{4}\right)\|v\|^2 \\
&\geq \left(c - \frac{b^2}{4}\right)\|v\|^2 \\
&> 0
\end{aligned}$$

via C.S.
$-|\langle u, v\rangle| \geq -\|u\|\,\|v\|$

Since $b^2 < 4c$, i.e.
$0 < 4c - b^2$
$0 < c - b^2/4$

So, $(T^2 + bT + cI)v \neq 0$ if $v \neq 0$.
Hence, it is injective & thus invertible.

7.27 *minimal polynomial of self-adjoint operator*

Suppose $T \in \mathcal{L}(V)$ is self-adjoint. Then the minimal polynomial of T equals $(z-\lambda_1)\cdots(z-\lambda_m)$ for some $\lambda_1, \ldots, \lambda_m \in \mathbf{R}$.

$\boxed{\text{proof}}$ First, if $\mathbb{F}=\mathbb{C}$, then the zeros of the min. poly. of T are the eigenvalues of T, all of which are real (as we proved last time). So, since polynomials factor completely over $\mathbb{C}$, the min. poly. of T is of the desired form.

Now, if $\mathbb{F}=\mathbb{R}$, then min. poly. of T is of the form $(z-\lambda_1)\cdots(z-\lambda_m)(z^2+b_1z+c_1)\cdots(z^2+b_nz+c_n)$ for some $\lambda_1,\ldots,\lambda_m \in \mathbb{R}$ & $b_1,c_1,\ldots,b_n,c_n \in \mathbb{R}$ s.t. $b_k^2 < 4c_k$ (since the roots of $z^2+b_kz+c_k$ are $z = \frac{-b_k \pm \sqrt{b_k^2-4c_k}}{2}$, which is real unless $b_k^2-4c_k<0$, i.e. $b_k^2<4c_k$). But this says

$$(T-\lambda_1I)\cdots(T-\lambda_mI)(T^2+b_1T+c_1I)\cdots(T^2+b_nT+c_nI)=0$$

and since $T^2+b_nT+c_nI$ is invertible, we get (if there are any such quadratic terms) that $(T-\lambda_1I)\cdots(T-\lambda_mI)(T^2+b_1T+c_1I)\cdots(T^2+b_{n-1}T+c_{n-1}I)=0$

This would contradict the minimality of the degree of the minimal polynomial. Hence, there must be no quadratic terms & the min. poly. must be of the form $(z-\lambda_1)\cdots(z-\lambda_m)$ for some $\lambda_1,\ldots,\lambda_m \in \mathbb{R}$.

7.29 *real spectral theorem*

Suppose $\mathbf{F} = \mathbf{R}$ and $T \in \mathcal{L}(V)$. Then the following are equivalent.

(a) T is self-adjoint.

(b) T has a diagonal matrix with respect to some orthonormal basis of V.

(c) V has an orthonormal basis consisting of eigenvectors of T.

So, self-adjoint is the same as diagonalizable by an orthonormal basis of eigenvectors

$\boxed{\text{Proof}}$ Suppose (a) is true. Then, T has an UT matrix w.r.t. an orthonormal basis of V since its min. poly. is of the form $(z-\lambda_1)\cdots(z-\lambda_m)$ for some $\lambda_1, \ldots, \lambda_m \in \mathbb{R}$ (this is an old result we proved in Ch 6). W.r.t. this basis, $\mathcal{M}(T)$ is U.T. &, since T is self adjoint, we get $\mathcal{M}(T) = \mathcal{M}(T^*) = \mathcal{M}(T)^t$. However, $\mathcal{M}(T)^t$ is lower triangular, while $\mathcal{M}(T)$ is UT, so $\mathcal{M}(T)$ must be diagonal. Therefore, $(a) \Rightarrow (b)$.

Now, suppose (b). Then, $\mathcal{M}(T)^t = \mathcal{M}(T^*) = \mathcal{M}(T)$ (since $\mathcal{M}(T) = \mathcal{M}(T)^t$) Hence, $T = T^*$ & $(b) \Rightarrow (a)$.

That $(c) \Leftrightarrow (b)$ was shown in Chapter 5 (i.e. diagonalizability of T $\Leftrightarrow \exists$ of a basis for V of e-vectors of T).

Complex Spectral Theorem

7.31 complex spectral theorem

Suppose $\mathbf{F} = \mathbf{C}$ and $T \in \mathcal{L}(V)$. Then the following are equivalent.

(a) T is normal.

(b) T has a diagonal matrix with respect to some orthonormal basis of V.

(c) V has an orthonormal basis consisting of eigenvectors of T.

$\boxed{\text{Proof}}$ Suppose (a). By Schur's theorem (from Ch 6), there is an orthonormal basis of V, $e_1, \ldots, e_n$, s.t. $\mathcal{M}(T)$ is upper triangular w.r.t. this basis. Let $\mathcal{M}(T) = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ & \ddots & \vdots \\ 0 & & a_{nn} \end{bmatrix}$.

Then, $Te_1 = a_{11}e_1$, so $\|Te_1\|^2 = |a_{11}|^2$ & since $\mathcal{M}(T^*) = \mathcal{M}(T)^*$ (where the second $*$ denotes the conjugate transpose) $T^*e_1 = \overline{a_{11}}e_1 + \cdots + \overline{a_{1n}}e_n$.

So, since $\|Tv\| = \|T^*v\|$ when T is normal, we get

$$|a_{11}|^2 = \|Te_1\|^2 = \|T^*e_1\|^2 = |a_{11}|^2 + \cdots + |a_{1n}|^2$$

Hence, $a_{12}, \ldots, a_{1n}$ are all 0. Repeating this process for $e_2, \ldots, e_n$ yields that the off-diagonal entries of $\mathcal{M}(T)$ are all 0. Hence, it is diagonal & (a)$\Rightarrow$(b).

Now, suppose (b). Then, T and T^* both have diagonal matrices w.r.t. some orthonormal basis of V (in fact, w.r.t. the same orthonormal basis of V). Since any two diagonal matrices commute, $\mathcal{M}(TT^*) = \mathcal{M}(T)\mathcal{M}(T^*) \overset{?}{=} \mathcal{M}(T^*)\mathcal{M}(T) = \mathcal{M}(T^*T)$. Therefore, $TT^* = T^*T$ & T commutes w/ its adjoint (that is, T is normal). So, (b)$\Rightarrow$(a).

(c)$\Leftrightarrow$(b) is the same as in the real case.