

7A Self-Adjoint and Normal Operators

Adjoint

7.1 definition: adjoint, T^*

Suppose $T \in \mathcal{L}(V, W)$. The adjoint of T is the function $T^*: W \rightarrow V$ such that

$$\langle Tv, w \rangle_W = \langle v, T^*w \rangle_V$$

for every $v \in V$ and every $w \in W$.

Idea How do we know such a T^* exists? Consider the linear functionals $\varphi_w \in W'$ given by $\varphi_w(u) = \langle u, w \rangle$. Then, $\varphi_w \circ T \in V'$, since the composition of linear maps is linear. Hence, by the Riesz rep theorem, there is some $v_w \in V$ s.t.
 $(\varphi_w \circ T)(v) = \langle v, v_w \rangle$, which says $\langle Tv, w \rangle = \langle v, v_w \rangle$.
So, $T^*w = v_w$ is how we define T^* . Now, is it true that $T^* \in \mathcal{L}(W, V)$? Yes!

7.4 adjoint of a linear map is a linear map

If $T \in \mathcal{L}(V, W)$, then $T^* \in \mathcal{L}(W, V)$.

proof $\S$ $T \in \mathcal{L}(V, W)$. Now, if $v \in V$ & $w_1, w_2 \in W$, then $\langle Tv, w_1 + w_2 \rangle = \langle Tv, w_1 \rangle + \langle Tv, w_2 \rangle$
 $= \langle v, T^*w_1 \rangle + \langle v, T^*w_2 \rangle$
 $= \langle v, T^*w_1 + T^*w_2 \rangle$

So, $\langle Tv, w_1 + w_2 \rangle = \langle v, T^*(w_1 + w_2) \rangle = \langle v, T^*w_1 + T^*w_2 \rangle$

and we may conclude (via the uniqueness ensured by Riesz rep) that

$$T^*(w_1 + w_2) = T^*w_1 + T^*w_2.$$

we do the same for homogeneity, taking $\lambda \in \mathbb{F}$ & noting that

$$\langle v, T^*(\lambda w) \rangle = \langle Tv, \lambda w \rangle = \bar{\lambda} \langle Tv, w \rangle = \bar{\lambda} \langle v, T^*w \rangle = \langle v, \lambda(T^*w) \rangle$$

So, $T^*(\lambda w) = \lambda T^*w$ & T^* is linear!

7.5 properties of the adjoint

Suppose $T \in \mathcal{L}(V, W)$. Then

- (a) $(S + T)^* = S^* + T^*$ for all $S \in \mathcal{L}(V, W)$; ✓
- (b) $(\lambda T)^* = \bar{\lambda} T^*$ for all $\lambda \in \mathbf{F}$; ✓
- (c) $(T^*)^* = T$; ✓
- (d) $(ST)^* = T^* S^*$ for all $S \in \mathcal{L}(W, U)$ (here U is a finite-dimensional inner product space over $\mathbf{F}$); ✓
- (e) $I^* = I$, where I is the identity operator on V ; ✓
- (f) if T is invertible, then T^* is invertible and $(T^*)^{-1} = (T^{-1})^*$. ✓

Proof

$$\begin{aligned}
 (a) \quad \langle (S+T)v, w \rangle &= \langle Sv + Tv, w \rangle \\
 &= \langle Sv, w \rangle + \langle Tv, w \rangle \\
 &= \langle v, S^*w \rangle + \langle v, T^*w \rangle \\
 &= \langle v, S^*w + T^*w \rangle \\
 &= \langle v, (S^* + T^*)w \rangle
 \end{aligned}$$

$$\text{So, } (S+T)^* = S^* + T^*.$$

$$\begin{aligned}
 (b) \quad \langle v, (\lambda T)^*w \rangle &= \langle (\lambda T)v, w \rangle = \lambda \langle Tv, w \rangle \\
 &= \lambda \langle v, T^*w \rangle \\
 &= \langle v, \bar{\lambda} T^*w \rangle \quad \text{So, } (\lambda T)^* = \bar{\lambda} T^*
 \end{aligned}$$

$$\begin{aligned}
 (c) \quad \langle Tv, w \rangle &= \langle v, T^*w \rangle = \overline{\langle T^*w, v \rangle} = \overline{\langle w, (T^*)^*v \rangle} = \langle (T^*)^*v, w \rangle \\
 \text{Hence, } Tv &= (T^*)^*v \text{ for all } v \in V, \text{ i.e. } T = (T^*)^*
 \end{aligned}$$

$$\begin{aligned}
 (d) \quad \langle v, (ST)^*w \rangle &= \langle STv, w \rangle = \langle Tv, S^*w \rangle = \langle v, T^*S^*w \rangle \\
 \text{So, } (ST)^* &= T^*S^*
 \end{aligned}$$

$$\begin{aligned}
 (e) \quad \langle u, I^*v \rangle &= \langle Iu, v \rangle = \langle u, v \rangle = \langle u, Iv \rangle \\
 \text{So, } I^* &= I
 \end{aligned}$$

$$\begin{aligned}
 (f) \quad \& \text{ } T \text{ is invertible. Then, } T^{-1}T = I \quad \& \quad (T^{-1}T)^* = T^*(T^{-1})^* = I^* = I \\
 \text{Hence, } T^* & \text{ is invertible } \& \quad (T^*)^{-1} = (T^{-1})^* \quad \text{Same!} \\
 (\text{Technically, we should also note that } TT^{-1} &= I \quad \& \quad (TT^{-1})^* = (T^{-1})^*T^* = I^* = I \text{ as well})
 \end{aligned}$$

7.6 null space and range of T^*

Suppose $T \in \mathcal{L}(V, W)$. Then

- (a) $\text{null } T^* = (\text{range } T)^\perp$;
- (b) $\text{range } T^* = (\text{null } T)^\perp$;
- (c) $\text{null } T = (\text{range } T^*)^\perp$;
- (d) $\text{range } T = (\text{null } T^*)^\perp$.

Proof (a) $w \in \text{null } T^* \Leftrightarrow T^*w = 0$
 $\Leftrightarrow \langle v, T^*w \rangle = 0$ for all $v \in V$
 $\Leftrightarrow \langle Tv, w \rangle = 0$ for all $v \in V$
 $\Leftrightarrow w \in (\text{range } T)^\perp$ (since it is orthogonal to every Tv)

(b) This is part (d) w/ T replaced by T^* , since $(T^*)^* = T$
(c) This is part (a) w/ T replaced by T^* , since $(T^*)^* = T$
(d) Using (a), $\text{null } T^* = (\text{range } T)^\perp$, so $(\text{null } T^*)^\perp = ((\text{range } T)^\perp)^\perp = \text{range } T$.

7.7 definition: conjugate transpose, A^*

The conjugate transpose of an m -by- n matrix A is the n -by- m matrix A^* obtained by interchanging the rows and columns and then taking the complex conjugate of each entry. In other words, if $j \in \{1, \dots, n\}$ and $k \in \{1, \dots, m\}$, then

$$(A^*)_{j,k} = \overline{A_{k,j}}.$$

7.9 matrix of T^* equals conjugate transpose of matrix of T

Let $T \in \mathcal{L}(V, W)$. Suppose $e_1, \dots, e_n$ is an orthonormal basis of V and $f_1, \dots, f_m$ is an orthonormal basis of W . Then $\mathcal{M}(T^*, (f_1, \dots, f_m), (e_1, \dots, e_n))$ is the conjugate transpose of $\mathcal{M}(T, (e_1, \dots, e_n), (f_1, \dots, f_m))$. In other words,

$$\mathcal{M}(T^*) = (\mathcal{M}(T))^*.$$

Proof
 (sketch)

$$Te_k = \langle Te_k, f_1 \rangle f_1 + \dots + \langle Te_k, f_m \rangle f_m$$

$$\text{So, } \mathcal{M}(T)_{j,k} = \langle Te_k, f_j \rangle = \langle e_k, T^* f_j \rangle = \overline{\langle T^* f_j, e_k \rangle} = \overline{\mathcal{M}(T^*)_{k,j}}$$

for the same reason, i.e.,

$$T^* f_j = \langle T^* f_j, e_1 \rangle e_1 + \dots + \langle T^* f_j, e_n \rangle e_n$$

Self-Adjoint Operators

7.10 definition: self-adjoint

An operator $T \in \mathcal{L}(V)$ is called self-adjoint if $T = T^*$.

An operator $T \in \mathcal{L}(V)$ is self-adjoint if and only if

$$\langle Tv, w \rangle = \langle v, Tw \rangle \quad \checkmark$$

for all $v, w \in V$.

Notice, if $T \in \mathcal{L}(V, W)$, then $T^* \in \mathcal{L}(W, V)$
So $T = T^*$ requires $V = W$

$$\begin{aligned} \text{i.e. } \langle Tv, w \rangle - \langle v, Tw \rangle &= 0 \\ \langle v, T^*w \rangle - \langle v, Tw \rangle &= 0 \\ \langle v, (T^* - T)w \rangle &= 0 \\ T^* - T &= 0 \\ T &= T^* \end{aligned}$$

7.12 eigenvalues of self-adjoint operators

Every eigenvalue of a self-adjoint operator is real.

Proof $\S$ $T \in \mathcal{L}(V)$ is self-adjoint. Let λ be an eigenvalue of T , and let $v \in V$ be a corresponding eigenvector (which means $v \neq 0$). Then,

$$\begin{aligned} \lambda \|v\|^2 &= \lambda \langle v, v \rangle = \langle \lambda v, v \rangle = \langle Tv, v \rangle = \langle v, Tv \rangle = \langle v, \lambda v \rangle = \bar{\lambda} \langle v, v \rangle = \bar{\lambda} \|v\|^2 \\ \text{So, } \lambda \|v\|^2 &= \bar{\lambda} \|v\|^2, \text{ and dividing by } \|v\|^2 \neq 0, \text{ we get } \lambda = \bar{\lambda}. \\ \text{So, } \lambda &\text{ is real!} \end{aligned}$$

$$\begin{aligned} \langle (T^* - T)v, w \rangle &= 0 \\ \langle (T - T^*)v, w \rangle &= 0 \end{aligned}$$

$$\begin{aligned} \text{So, } \langle (T - T^*)v, v \rangle &= 0 \\ \S T - T^* &= 0 \\ \text{or } T &= T^* \end{aligned}$$

7.13 Tv is orthogonal to v for all $v \iff T = 0$ (assuming $F = \mathbb{C}$)

Suppose V is a complex inner product space and $T \in \mathcal{L}(V)$. Then

$$\langle Tv, v \rangle = 0 \text{ for every } v \in V \iff T = 0.$$

proof If $u, w \in V$, then

$$\begin{aligned} \langle Tu, w \rangle &= \frac{\langle T(u+w), u+w \rangle - \langle T(u-w), u-w \rangle}{4} + \frac{\langle T(u+iw), u+iw \rangle - \langle T(u-iw), u-iw \rangle}{4} \quad \text{---} \\ &= \frac{\langle Tu + Tw, u + w \rangle - \langle Tu - Tw, u - w \rangle}{4} + \frac{\langle Tu + iTw, u + iw \rangle - \langle Tu - iTw, u - iw \rangle}{4} \\ &= \frac{\langle Tu, u \rangle + \langle Tu, w \rangle + \langle Tw, u \rangle + \langle Tw, w \rangle - \langle Tu, u \rangle + \langle Tu, w \rangle + \langle Tw, u \rangle - \langle Tw, w \rangle}{4} \\ &\quad + \frac{\langle Tu, u \rangle + \langle Tu, w \rangle - \langle Tw, u \rangle + \langle Tw, w \rangle - \langle Tu, u \rangle + \langle Tu, w \rangle - \langle Tw, u \rangle - \langle Tw, w \rangle}{4} \\ &= \frac{4\langle Tu, w \rangle}{4} = \langle Tu, w \rangle \end{aligned}$$

The RHS is of the form $\langle Tv, w \rangle$ in each term, so $\langle Tu, w \rangle = 0 \forall u, w \in V$.
Hence, $Tu = 0$
 $\S T = 0$.