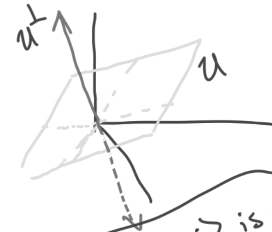
6C Orthogonal Complements and Minimization Problems



6.46 definition: *orthogonal complement, $U^\perp$*

If U is a subset of V, then the *orthogonal complement* of U, denoted by $U^\perp$, is the set of all vectors in V that are orthogonal to every vector in U:

$$U^\perp = \{v \in V : \langle u, v\rangle = 0 \text{ for every } u \in U\}.$$

Since $\langle \cdot, v\rangle$ is a linear functional on V, all we are doing is taking $v \in V$ s.t. $U \subseteq \operatorname{null}(\langle \cdot, v\rangle)$

6.48 *properties of orthogonal complement*

(a) If U is a subset of V, then $U^\perp$ is a subspace of V.

(b) $\{0\}^\perp = V$.

(c) $V^\perp = \{0\}$.

(d) If U is a subset of V, then $U \cap U^\perp \subseteq \{0\}$.

(e) If G and H are subsets of V and $G \subseteq H$, then $H^\perp \subseteq G^\perp$.

$\boxed{\text{Proof}}$ (a) If $v \in U^\perp$, then $\langle u, v\rangle = 0$ for all $u \in U$. So, $\langle u, \lambda v\rangle = \bar{\lambda}\langle u, v\rangle = \bar{\lambda}(0) = 0$ for all $u \in U$ and $\lambda v \in U^\perp$ as well for any $\lambda \in \mathbb{F}$. Similarly, if $w \in U^\perp$, then $\langle u, v+w\rangle = \langle u, v\rangle + \langle u, w\rangle = 0 + 0 = 0$. So, $v + w \in U^\perp$ as well. Clearly, $\langle u, 0\rangle = 0$, so $0 \in U^\perp$. Hence, $U^\perp \leq V$.

(b) $\{0\}^\perp = \{v \in V : \langle 0, v\rangle = 0\} = V$ (by definition of $\langle \cdot, \cdot \rangle$)

(c) $V^\perp = \{v \in V : \langle w, v\rangle = 0 \ \forall w \in V\} \subseteq \{v \in V : \langle v, v\rangle = 0\} = \{v \in V : \|v\|^2 = 0\} = \{0\}$

(d) $U \cap U^\perp = U \cap \{v \in V : \langle u, v\rangle = 0 \ \forall u \in U\} = \{v \in U : \langle u, v\rangle = 0 \ \forall u \in U\} \subseteq \{v \in U : \langle v, v\rangle = 0\} \subseteq \{0\}$.

(e) If $G \subseteq H$, then $H^\perp = \{v \in V : \langle h, v\rangle = 0 \ \forall h \in H\} \subseteq \{v \in V : \langle g, v\rangle = 0 \ \forall g \in G\} = G^\perp$

6.49 *direct sum of a subspace and its orthogonal complement*

Suppose U is a finite-dimensional subspace of V. Then

$$V = U \oplus U^{\perp}.$$

$\boxed{\text{Proof}}$ Let $v \in V$ & $e_1, \dots, e_n$ is an ONB of U. Then,

$$v = v + 0 = v + (\langle v, e_1\rangle e_1 + \dots + \langle v, e_n\rangle e_n) - (\langle v, e_1\rangle e_1 + \dots + \langle v, e_n\rangle e_n)$$

Let $w = v - \overbrace{(\langle v, e_1\rangle e_1 + \dots + \langle v, e_n\rangle e_n)}^{u}$ & let $u = \overbrace{\langle v, e_1\rangle e_1 + \dots + \langle v, e_n\rangle e_n}^{P_U v}$ So that $v = u + w$. Certainly, $u \in U$, since it is a linear combo of a basis for U. Moreover,

$$\begin{aligned}
\langle w, e_k\rangle &= \langle v - u, e_k\rangle \\
&= \langle v, e_k\rangle - \langle u, e_k\rangle \\
&= \langle v, e_k\rangle - \langle \langle v, e_1\rangle e_1 + \dots + \langle v, e_n\rangle e_n, e_k\rangle \\
&= \langle v, e_k\rangle - \left(\langle v, e_1\rangle \langle e_1, e_k\rangle + \dots + \langle v, e_n\rangle \langle e_n, e_k\rangle\right) \\
&= \langle v, e_k\rangle - \left(\langle v, e_k\rangle \langle e_k, e_k\rangle\right) \\
&= \langle v, e_k\rangle - \langle v, e_k\rangle \\
&= 0
\end{aligned}$$

So, $w \in U^{\perp}$, since w is orthogonal to every element in a basis of U. Hence, $V = U + U^{\perp}$. Moreover, this sum is direct since $U \cap U^{\perp} = \{0\}$ by the previous theorem.

6.51 *dimension of orthogonal complement*

Suppose V is finite-dimensional and U is a subspace of V. Then

$$\dim U^{\perp} = \dim V - \dim U.$$

6.52 *orthogonal complement of the orthogonal complement*

Suppose U is a finite-dimensional subspace of V. Then

$$U = (U^\perp)^\perp.$$

$\boxed{\text{Proof}}$ Notice $U^\perp = \{v \in V : \langle u, v\rangle = 0 \ \forall u \in U\}$

So, $(U^\perp)^\perp = \{v \in V : \langle w, v\rangle = 0 \ \forall w \in U^\perp\}$

but notice that if $u \in U$, then $\langle w, u\rangle = \overline{\langle u, w\rangle} = \bar{0} = 0$ whenever $w \in U^\perp$. So, $U \subseteq (U^\perp)^\perp$.

For the reverse inclusion, $\exists\, v \in (U^\perp)^\perp$. By the previous result, $v = u + w$ where $u \in U$ & $w \in U^\perp$. So, $v - u = w \in U^\perp$. So, $v - u \in U^\perp$, and since $U \subseteq (U^\perp)^\perp$, $v - u \in (U^\perp)^\perp$. Hence, $v - u \in U^\perp \cap (U^\perp)^\perp = \{0\}$, hence $v - u = 0$ & $v = u \in U$. So $(U^\perp)^\perp \subseteq U$. Therefore, $U = (U^\perp)^\perp$.

6.54 $U^\perp = \{0\} \iff U = V$ *(for U a finite-dimensional subspace of V)*

Suppose U is a finite-dimensional subspace of V. Then

$$U^\perp = \{0\} \iff U = V.$$

$\boxed{\text{Proof}}$

$$U^\perp = \{0\} \iff (U^\perp)^\perp = \{0\}^\perp$$

$$\iff U = V \quad \left(\text{since } (U^\perp)^\perp = U \text{ \& } \{0\}^\perp = V\right.$$

6.55 definition: *orthogonal projection, P_U*

Suppose U is a finite-dimensional subspace of V. The *orthogonal projection* of V onto U is the operator $P_U \in \mathcal{L}(V)$ defined as follows: For each $v \in V$, write $v = u + w$, where $u \in U$ and $w \in U^\perp$. Then let $P_U v = u$.

For an ONB $e_1, \dots, e_n$ of U, $P_U v$ is just

$P_U v = \langle v, e_1 \rangle e_1 + \dots + \langle v, e_n \rangle e_n$

6.57 *properties of orthogonal projection P_U*

Suppose U is a finite-dimensional subspace of V. Then

(a) $P_U \in \mathcal{L}(V)$; ← easy!

(b) $P_U u = u$ for every $u \in U$; ← directly from this

(c) $P_U w = 0$ for every $w \in U^\perp$; ← easy, from definition of P_U

(d) range $P_U = U$; Use (b) & (c) w/ $V = U \oplus U^\perp$

(e) null $P_U = U^\perp$; Same logic!

(f) $v - P_U v \in U^\perp$ for every $v \in V$; ← $v = u + w$, $u \in U$, $w \in U^\perp$; $v = P_U v + w \Rightarrow v - P_U v = w \in U^\perp$

(g) $P_U^2 = P_U$; If $v = u + w$, $u \in U$, $w \in U^\perp$, $P_U v = u$ & $P_U^2 v = P_U u = u$

(h) $\|P_U v\| \le \|v\|$ for every $v \in V$; $\|u\| \le \|u + w\| = \|v\|$ since u & w are orthogonal

(i) if $e_1, \dots, e_m$ is an orthonormal basis of U and $v \in V$, then

$$P_U v = \langle v, e_1 \rangle e_1 + \dots + \langle v, e_m \rangle e_m.$$

$\boxed{\text{Proof}}$ (h) Since $V = U \oplus U^\perp$, write $v \in V$ as $v = u + w$ where $u \in U$ & $w \in U^\perp$. Since $P_U v = u$, and since u & w are orthogonal, we get $\|P_U v\|^2 = \|u\|^2 \le \|u\|^2 + \|w\|^2 = \|u + w\|^2$ via the Pythagorean theorem. Taking a square root, $\|P_U v\| \le \|u + w\| = \|v\|$.

(i) If $u \in U$, then $u = \langle u, e_1 \rangle e_1 + \dots + \langle u, e_m \rangle e_m$ for any ONB $e_1, \dots, e_m$ of U by a previous result. So, since $P_U v = P_U(u + w) = u$, we get that $\underline{P_U v = \langle u, e_1 \rangle e_1 + \dots + \langle u, e_m \rangle e_m}$

But notice: since $w \in U^\perp$, $\langle w, e_k \rangle = 0$, so

$$\begin{aligned} P_U v &= \langle u, e_1 \rangle e_1 + \dots + \langle u, e_m \rangle e_m \\ &= (\langle u, e_1 \rangle + 0) e_1 + \dots + (\langle u, e_m \rangle + 0) e_m \\ &= (\langle u, e_1 \rangle + \langle w, e_1 \rangle) e_1 + \dots + (\langle u, e_m \rangle + \langle w, e_m \rangle) e_m \\ &= \langle v, e_1 \rangle e_1 + \dots + \langle v, e_m \rangle e_m. \end{aligned}$$

6.58 *Riesz representation theorem, revisited*

Suppose V is finite-dimensional. For each $v \in V$, define $\varphi_v \in V'$ by

$$\varphi_v(u) = \langle u, v \rangle$$

for each $u \in V$. Then $v \mapsto \varphi_v$ is a one-to-one function from V onto V'.

$\boxed{\text{Note}}$ If $\mathbb{F} = \mathbb{R}$, then $\varphi_{\lambda v}(u) = \langle u, \lambda v \rangle = \bar{\lambda} \langle u, v \rangle = \lambda \langle u, v \rangle = \lambda \varphi_v(u)$

So, in the real case, the map $T(v) = \varphi_v$ is linear! i.e. $T \in \mathcal{L}(V, V')$ if V is a real inner product space. But, if $\mathbb{F} = \mathbb{C}$, then T is not linear.

$\boxed{\text{Proof}}$ (if $\mathbb{F} = \mathbb{R}$) Let $T \in \mathcal{L}(V, V')$ be $Tv = \varphi_v$. Then,

Notice that $\operatorname{null} T = \{v \in V : Tv = 0 \in V'\} = \{v \in V : \langle u, v \rangle = 0 \ \forall u \in V\} \subseteq \{v \in V : \langle v, v \rangle = 0\} = \{0\}$

Hence, $\operatorname{null} T = \{0\}$ & T is injective. Since $\dim V = \dim V'$, T is then also surjective & an isomorphism.

(If $\mathbb{F} = \mathbb{C}$) In this case, the above works to show that the set of vectors $v \in V$ s.t. $\varphi_v = 0$ is just $\{0\}$.

For surjectivity, all we need to do is note that $\varphi_0 = 0$ & for $v \in V$, $v \neq 0$, $\operatorname{null} \varphi_v$ has dimension $\dim V - 1$. So, $(\operatorname{null} \varphi_v)^\perp$ is one dimensional. The same is true of any $\varphi \in V'$, so all we need to do is "match up" the null spaces' orthogonal complements & scale appropriately. For details, see page 216 & 217 in the book.