

6B Orthonormal Bases

Orthonormal Lists and the Gram-Schmidt Procedure

6.22 definition: orthonormal

ortho - orthogonal, pairwise
normal - "normalized" i.e. norm 1

- A list of vectors is called orthonormal if each vector in the list has norm 1 and is orthogonal to all the other vectors in the list.
- In other words, a list $e_1, \dots, e_m$ of vectors in V is orthonormal if

$$\langle e_j, e_k \rangle = \begin{cases} 1 & \text{if } j = k, \\ 0 & \text{if } j \neq k \end{cases}$$

so, $\langle e_j, e_j \rangle = 1 = \|e_j\|^2 \Rightarrow \|e_j\| = 1$

for all $j, k \in \{1, \dots, m\}$.

6.24 norm of an orthonormal linear combination

Suppose $e_1, \dots, e_m$ is an orthonormal list of vectors in V . Then

$$\|a_1 e_1 + \dots + a_m e_m\|^2 = \underbrace{|a_1|^2 + \dots + |a_m|^2}_{\text{sum of squares}}$$

for all $a_1, \dots, a_m \in \mathbb{F}$.

Proof Using the pythagorean theorem (which, recall, says that $\|v+w\|^2 = \|v\|^2 + \|w\|^2$ for v, w orthogonal), we get

$$\begin{aligned} \|a_1 e_1 + \dots + a_m e_m\|^2 &= \|a_1 e_1\|^2 + \|a_2 e_2 + \dots + a_m e_m\|^2 \\ &\vdots \\ &= \|a_1 e_1\|^2 + \dots + \|a_m e_m\|^2 \\ &= |a_1|^2 \|e_1\|^2 + \dots + |a_m|^2 \|e_m\|^2 \\ &= |a_1|^2 + \dots + |a_m|^2 \end{aligned}$$

Since $\langle a_1 e_1, a_2 e_2 + \dots + a_m e_m \rangle = a_1(a_2 \langle e_1, e_2 \rangle + \dots + a_m \langle e_1, e_m \rangle) = a_1(a_2(0) + \dots + a_m(0)) = 0$

6.25 orthonormal lists are linearly independent

Every orthonormal list of vectors is linearly independent.

Proof Let $e_1, \dots, e_m$ be an ON list in V . Let $a_1, \dots, a_m \in \mathbb{F}$ be a list of scalars s.t. $a_1 e_1 + \dots + a_m e_m = 0$. Then, using the previous result, $0 = \|a_1 e_1 + \dots + a_m e_m\|^2 = |a_1|^2 + \dots + |a_m|^2$.
So, $\underline{a_1 = \dots = a_m = 0}$ and $e_1, \dots, e_m$ is lin. indep.

6.26 Bessel's inequality

Suppose $e_1, \dots, e_m$ is an orthonormal list of vectors in V . If $v \in V$ then

$$|\langle v, e_1 \rangle|^2 + \dots + |\langle v, e_m \rangle|^2 \leq \|v\|^2.$$

Proof Set $u = \langle v, e_1 \rangle e_1 + \dots + \langle v, e_m \rangle e_m$. Let $w = v - u$. Then, $v = u + w$. Now, if $\langle w, u \rangle = 0$, then by the Pythagorean theorem, we have $\|v\|^2 = \|u\|^2 + \|w\|^2 \geq \|u\|^2 = |\langle v, e_1 \rangle|^2 + \dots + |\langle v, e_m \rangle|^2$ by the previous result. So, we need only show $\langle w, u \rangle = 0$. To wit,

$$\begin{aligned} \langle w, e_k \rangle &= \langle v - u, e_k \rangle = \langle v - (\langle v, e_1 \rangle e_1 + \dots + \langle v, e_m \rangle e_m), e_k \rangle \\ &= \langle v, e_k \rangle - (\langle v, e_1 \rangle \langle e_1, e_k \rangle + \dots + \langle v, e_m \rangle \langle e_m, e_k \rangle) \\ &= \langle v, e_k \rangle - \langle v, e_k \rangle \langle e_k, e_k \rangle \\ &= \langle v, e_k \rangle - \langle v, e_k \rangle \|e_k\|^2 \\ &= \langle v, e_k \rangle - \langle v, e_k \rangle \\ &= 0 \end{aligned}$$

$$\begin{aligned} \text{Hence, } \langle w, u \rangle &= \langle w, \langle v, e_1 \rangle e_1 + \dots + \langle v, e_m \rangle e_m \rangle \\ &= \overline{\langle v, e_1 \rangle} \langle w, e_1 \rangle + \dots + \overline{\langle v, e_m \rangle} \langle w, e_m \rangle \\ &= \overline{\langle v, e_1 \rangle} (0) + \dots + \overline{\langle v, e_m \rangle} (0) \\ &= 0. \end{aligned}$$

6.27 definition: orthonormal basis

An orthonormal basis of V is an orthonormal list of vectors in V that is also a basis of V .

6.28 orthonormal lists of the right length are orthonormal bases

Suppose V is finite-dimensional. Then every orthonormal list of vectors in V of length $\dim V$ is an orthonormal basis of V .

Proof If $e_1, \dots, e_{\dim V}$ is orthonormal, then it is a lin. indep. list of length $\dim V$ & is therefore a basis.

6.30 writing a vector as a linear combination of an orthonormal basis

Suppose $e_1, \dots, e_n$ is an orthonormal basis of V and $u, v \in V$. Then

(a) $\underline{v} = \langle v, e_1 \rangle e_1 + \dots + \langle v, e_n \rangle e_n$, ✓

(b) $\|v\|^2 = |\langle v, e_1 \rangle|^2 + \dots + |\langle v, e_n \rangle|^2$,

(c) $\langle u, v \rangle = \langle u, e_1 \rangle \overline{\langle v, e_1 \rangle} + \dots + \langle u, e_n \rangle \overline{\langle v, e_n \rangle}$.

Proof (a) Since $e_1, \dots, e_n$ is an ONB, there exist scalars $a_1, \dots, a_n \in \mathbb{F}$ s.t. $v = a_1 e_1 + \dots + a_n e_n$. But, notice:

$$\begin{aligned} \langle v, e_k \rangle &= \langle a_1 e_1 + \dots + a_n e_n, e_k \rangle \\ &= a_1 \langle e_1, e_k \rangle + \dots + a_n \langle e_n, e_k \rangle \\ &= a_k \langle e_k, e_k \rangle \\ &= a_k \end{aligned}$$

Hence, $v = \langle v, e_1 \rangle e_1 + \dots + \langle v, e_n \rangle e_n$.

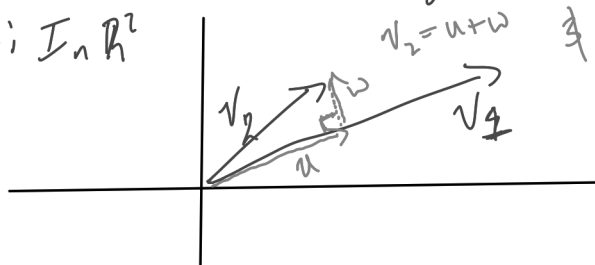
(b) Using (a), $\|v\|^2 = |\langle v, e_1 \rangle|^2 + \dots + |\langle v, e_n \rangle|^2$ via the first theorem of this section.

(c) Using part (a), we get

$$\begin{aligned} \langle u, v \rangle &= \langle \langle u, e_1 \rangle e_1 + \dots + \langle u, e_n \rangle e_n, \langle v, e_1 \rangle e_1 + \dots + \langle v, e_n \rangle e_n \rangle \\ &= \langle \langle u, e_1 \rangle e_1, \langle v, e_1 \rangle e_1 + \dots + \langle v, e_n \rangle e_n \rangle + \dots + \langle \langle u, e_n \rangle e_n, \langle v, e_1 \rangle e_1 + \dots + \langle v, e_n \rangle e_n \rangle \\ &= \langle u, e_1 \rangle (\langle e_1, \langle v, e_1 \rangle e_1 \rangle + \dots + \langle e_1, \langle v, e_n \rangle e_n \rangle) + \dots + \langle u, e_n \rangle (\langle e_n, \langle v, e_1 \rangle e_1 \rangle + \dots + \langle e_n, \langle v, e_n \rangle e_n \rangle) \\ &= \langle u, e_1 \rangle (\overline{\langle v, e_1 \rangle} \langle e_1, e_1 \rangle + \dots + \overline{\langle v, e_n \rangle} \langle e_1, e_n \rangle) + \dots + \langle u, e_n \rangle (\overline{\langle v, e_1 \rangle} \langle e_n, e_1 \rangle + \dots + \overline{\langle v, e_n \rangle} \langle e_n, e_n \rangle) \\ &= \langle u, e_1 \rangle \overline{\langle v, e_1 \rangle} + \dots + \langle u, e_n \rangle \overline{\langle v, e_n \rangle} \end{aligned}$$

So, we are done!

Idea If we have a basis of an inner product space V , then how can we make that basis orthogonal in the easiest way? That is, visualize this: In $\mathbb{R}^2$



$$\nexists \langle w, v_1 \rangle = 0$$

So, just take v_2 and remove the part that isn't orthogonal to v_1 . Then, normalize! i.e. replace v_1, w by $\frac{v_1}{\|v_1\|}, \frac{w}{\|w\|}$

6.32 Gram-Schmidt procedure

Suppose $\underline{v}_1, \dots, \underline{v}_m$ is a linearly independent list of vectors in V . Let $\underline{f}_1 = \underline{v}_1$. For $k = 2, \dots, m$, define $\underline{f}_k$ inductively by

$$\underline{f}_k = \underline{v}_k - \frac{\langle \underline{v}_k, \underline{f}_1 \rangle}{\|\underline{f}_1\|^2} \underline{f}_1 - \dots - \frac{\langle \underline{v}_k, \underline{f}_{k-1} \rangle}{\|\underline{f}_{k-1}\|^2} \underline{f}_{k-1}.$$

For each $k = 1, \dots, m$, let $\underline{e}_k = \frac{\underline{f}_k}{\|\underline{f}_k\|}$. Then $\underline{e}_1, \dots, \underline{e}_m$ is an orthonormal list of vectors in V such that

$$\underline{\text{span}}(\underline{v}_1, \dots, \underline{v}_k) = \underline{\text{span}}(\underline{e}_1, \dots, \underline{e}_k)$$

for each $k = 1, \dots, m$.

Proof (Sketch)

Do induction on k . If $k=1$, then $\underline{f}_1 = \underline{v}_1$ & $\underline{e}_1 = \frac{\underline{f}_1}{\|\underline{f}_1\|} = \frac{\underline{v}_1}{\|\underline{v}_1\|}$. So, $\|\underline{e}_1\| = \|\frac{\underline{v}_1}{\|\underline{v}_1\|}\| = \frac{1}{\|\underline{v}_1\|} \|\underline{v}_1\| = 1$, and $0 \neq \underline{e}_1 \in \text{span}(\underline{v}_1)$ so $\text{span}(\underline{e}_1) = \text{span}(\underline{v}_1)$.

Then, & everything works up to $k-1$, i.e. that $\underline{e}_1, \dots, \underline{e}_{k-1}$ is an ON list st. $\text{span}(\underline{v}_1, \dots, \underline{v}_{k-1}) = \text{span}(\underline{e}_1, \dots, \underline{e}_{k-1})$. Then, build $\underline{e}_k$ & show everything is still good.

$$\underline{f}_k = \underline{v}_k - \frac{\langle \underline{v}_k, \underline{f}_1 \rangle}{\|\underline{f}_1\|^2} \underline{f}_1 - \dots - \frac{\langle \underline{v}_k, \underline{f}_{k-1} \rangle}{\|\underline{f}_{k-1}\|^2} \underline{f}_{k-1}$$

← $\underline{v}_k = \dots + \underline{f}_k$
& $\underline{v}_k$ is the in $\text{span}(\underline{e}_1, \dots, \underline{e}_k)$

$$\underline{e}_k = \frac{\underline{f}_k}{\|\underline{f}_k\|}$$

$$\text{So, } \langle \underline{e}_k, \underline{e}_j \rangle = \left\langle \frac{\underline{f}_k}{\|\underline{f}_k\|}, \frac{\underline{f}_j}{\|\underline{f}_j\|} \right\rangle \quad (\text{where } j < k)$$

$$= \frac{1}{\|\underline{f}_k\| \|\underline{f}_j\|} \langle \underline{f}_k, \underline{f}_j \rangle$$

$$= \frac{1}{\|\underline{f}_k\| \|\underline{f}_j\|} \left\langle \underline{v}_k - \frac{\langle \underline{v}_k, \underline{f}_1 \rangle}{\|\underline{f}_1\|^2} \underline{f}_1 - \dots - \frac{\langle \underline{v}_k, \underline{f}_{k-1} \rangle}{\|\underline{f}_{k-1}\|^2} \underline{f}_{k-1}, \underline{f}_j \right\rangle$$

$$= \frac{1}{\|\underline{f}_k\| \|\underline{f}_j\|} \left(\langle \underline{v}_k, \underline{f}_j \rangle - \frac{\langle \underline{v}_k, \underline{f}_1 \rangle}{\|\underline{f}_1\|^2} \langle \underline{f}_1, \underline{f}_j \rangle - \dots - \frac{\langle \underline{v}_k, \underline{f}_{k-1} \rangle}{\|\underline{f}_{k-1}\|^2} \langle \underline{f}_{k-1}, \underline{f}_j \rangle \right)$$

$$= \frac{1}{\|\underline{f}_k\| \|\underline{f}_j\|} \left(\langle \underline{v}_k, \underline{f}_j \rangle - \frac{\langle \underline{v}_k, \underline{f}_j \rangle}{\|\underline{f}_j\|^2} \langle \underline{f}_j, \underline{f}_j \rangle \right)$$

$$= \frac{1}{\|\underline{f}_k\| \|\underline{f}_j\|} (\langle \underline{v}_k, \underline{f}_j \rangle - \langle \underline{v}_k, \underline{f}_j \rangle) = 0.$$

Then, show the span bit, which is not bad.

6.35 existence of orthonormal basis

Every finite-dimensional inner product space has an orthonormal basis.

Proof Take a basis & apply GS process.

6.36 every orthonormal list extends to an orthonormal basis

Suppose V is finite-dimensional. Then every orthonormal list of vectors in V can be extended to an orthonormal basis of V .

Proof Take the ON list, which is lin. indep., and extend to a basis. Then, apply GS to this basis. Only the elements added in the extension are changed, since the first part is already ON.

6.37 upper-triangular matrix with respect to some orthonormal basis

Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Then T has an upper-triangular matrix with respect to some orthonormal basis of V if and only if the minimal polynomial of T equals $(z - \lambda_1) \cdots (z - \lambda_m)$ for some $\lambda_1, \dots, \lambda_m \in \mathbb{F}$.

Proof This is the same condition as what we had for getting an UT matrix of T w/ a regular basis, so just use the fact that we can orthonormalize this basis w/ GS & we are done (using $\text{span}(v_1, \dots, v_k) = \text{span}(e_1, \dots, e_k)$).

6.38 Schur's theorem

Every operator on a finite-dimensional complex inner product space has an upper-triangular matrix with respect to some orthonormal basis.