

6A Inner Products and Norms

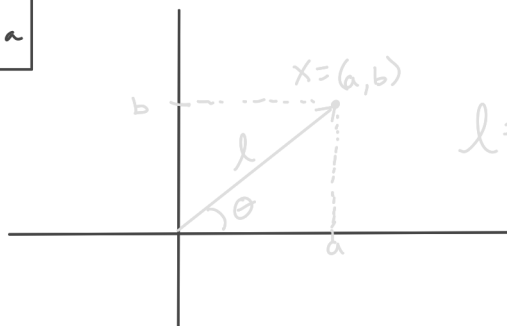
6.1 definition: dot product

For $\underline{x}, \underline{y} \in \mathbb{R}^n$, the dot product of $\underline{x}$ and $\underline{y}$, denoted by $\underline{x} \cdot \underline{y}$, is defined by

$$\underline{x} \cdot \underline{y} = x_1 y_1 + \cdots + x_n y_n,$$

where $\underline{x} = (x_1, \dots, x_n)$ and $\underline{y} = (y_1, \dots, y_n)$.

Idea



$$l = \sqrt{b^2 + a^2} = \sqrt{bb + aa} = \sqrt{\underline{x} \cdot \underline{x}} \leftarrow$$

$$\tan \theta = \frac{b}{a}$$

$$\theta = \tan^{-1}\left(\frac{b}{a}\right)$$

6.2 definition: inner product

An inner product on V is a function that takes each ordered pair $(\underline{u}, \underline{v})$ of elements of V to a number $\langle \underline{u}, \underline{v} \rangle \in \mathbb{F}$ and has the following properties.

positivity

$$\langle \underline{v}, \underline{v} \rangle \geq 0 \text{ for all } \underline{v} \in V.$$

definiteness

$$\langle \underline{v}, \underline{v} \rangle = 0 \text{ if and only if } \underline{v} = 0.$$

additivity in first slot

$$\langle \underline{u} + \underline{v}, \underline{w} \rangle = \langle \underline{u}, \underline{w} \rangle + \langle \underline{v}, \underline{w} \rangle \text{ for all } \underline{u}, \underline{v}, \underline{w} \in V.$$

homogeneity in first slot

$$\langle \lambda \underline{u}, \underline{v} \rangle = \lambda \langle \underline{u}, \underline{v} \rangle \text{ for all } \lambda \in \mathbb{F} \text{ and all } \underline{u}, \underline{v} \in V.$$

conjugate symmetry

$$\langle \underline{u}, \underline{v} \rangle = \overline{\langle \underline{v}, \underline{u} \rangle} \text{ for all } \underline{u}, \underline{v} \in V.$$

here, $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$

$$\text{i.e. } \langle \cdot, \cdot \rangle: V \times V \rightarrow \mathbb{F}$$

so, $\langle \underline{v}, \underline{v} \rangle > 0$ for $\underline{v} \neq 0$

So, for fixed $\underline{w} \in V$,
the function
 $\underline{v} \mapsto \langle \underline{v}, \underline{w} \rangle$ is a
linear functional

Recall: for $z = x + iy \in \mathbb{C}$,
 $\bar{z} = x - iy \in \mathbb{C}$
(i.e. throw a minus on
the imaginary part)

6.4 definition: inner product space

An inner product space is a vector space V along with an inner product on V .

6.6 basic properties of an inner product

- (a) For each fixed $\underline{v} \in V$, the function that takes $\underline{u} \in V$ to $\langle \underline{u}, \underline{v} \rangle$ is a linear map from V to $\mathbf{F}$.
- (b) $\langle \underline{0}, \underline{v} \rangle = \underline{0}$ for every $\underline{v} \in V$.
- (c) $\langle \underline{v}, \underline{0} \rangle = \underline{0}$ for every $\underline{v} \in V$.
- (d) $\langle \underline{u}, \underline{v} + \underline{w} \rangle = \langle \underline{u}, \underline{v} \rangle + \langle \underline{u}, \underline{w} \rangle$ for all $\underline{u}, \underline{v}, \underline{w} \in V$.
- (e) $\langle \underline{u}, \underline{\lambda v} \rangle = \overline{\lambda} \langle \underline{u}, \underline{v} \rangle$ for all $\lambda \in \mathbf{F}$ and all $\underline{u}, \underline{v} \in V$.

Proof

- (a) Since $\langle \underline{u} + \underline{w}, \underline{v} \rangle = \langle \underline{u}, \underline{v} \rangle + \langle \underline{w}, \underline{v} \rangle$ & $\langle \lambda \underline{u}, \underline{v} \rangle = \lambda \langle \underline{u}, \underline{v} \rangle$, we are done.
- (b) This is true for any linear map, so we get (b) from (a)
- (c) Using conjugate symmetry, $\langle \underline{v}, \underline{0} \rangle = \overline{\langle \underline{0}, \underline{v} \rangle} = \overline{0} = 0$.
- (d) $\langle \underline{u}, \underline{v} + \underline{w} \rangle = \overline{\langle \underline{v} + \underline{w}, \underline{u} \rangle} = \overline{\langle \underline{v}, \underline{u} \rangle + \langle \underline{w}, \underline{u} \rangle} = \overline{\langle \underline{v}, \underline{u} \rangle} + \overline{\langle \underline{w}, \underline{u} \rangle} = \langle \underline{u}, \underline{v} \rangle + \langle \underline{u}, \underline{w} \rangle$
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6.7 definition: norm, $\|v\|$

For $v \in V$, the norm of v , denoted by $\|v\|$, is defined by

$$\|v\| = \sqrt{\langle v, v \rangle}.$$

6.9 basic properties of the norm

Suppose $v \in V$.

(a) $\|v\| = 0$ if and only if $v = 0$. $\leftarrow$

(b) $\|\lambda v\| = |\lambda| \|v\|$ for all $\lambda \in \mathbb{F}$. $\leftarrow$

Proof (a) $\|v\| = 0 \Leftrightarrow \sqrt{\langle v, v \rangle} = 0$
 $\Leftrightarrow \langle v, v \rangle = 0$
 $\Leftrightarrow v = 0$

(b) $\|\lambda v\| = \sqrt{\langle \lambda v, \lambda v \rangle}$
 $= \sqrt{\lambda \langle v, \lambda v \rangle}$
 $= \sqrt{\lambda \overline{\lambda} \langle v, v \rangle}$
 $= \sqrt{|\lambda|^2 \langle v, v \rangle}$
 $= |\lambda| \sqrt{\langle v, v \rangle}$
 $= |\lambda| \|v\|$

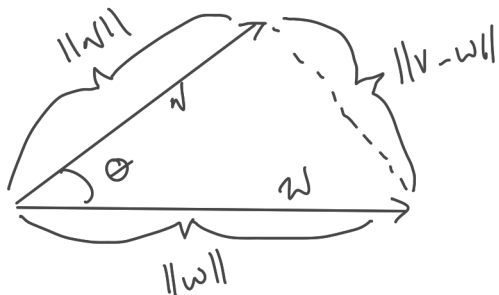
$z = x + iy$
 $\bar{z} = x - iy$
 $z \bar{z} = (x + iy)(x - iy)$
 $= x^2 + ixy - ixy - i^2 y^2$
 $= x^2 + y^2$
 $= |z|^2$

6.10 definition: orthogonal

Two vectors $u, v \in V$ are called orthogonal if $\langle u, v \rangle = 0$.

i.e. if they are at a right angle, using

Fact $\langle v, w \rangle = \|v\| \|w\| \cos \theta$



using the law of cosines,

$$\|v - w\|^2 = \|w\|^2 + \|v\|^2 - 2\|v\| \|w\| \cos \theta$$

$$\langle v - w, v - w \rangle = \|w\|^2 + \|v\|^2 - 2\|v\| \|w\| \cos \theta$$

$$\cancel{\|v\|^2} + \cancel{\|w\|^2} - 2\langle v, w \rangle = \cancel{\|w\|^2} + \cancel{\|v\|^2} - 2\|v\| \|w\| \cos \theta$$

$$\langle v, w \rangle = \|v\| \|w\| \cos \theta$$

Note: this calc assumes v, w are not scalar mult's of each other

6.11 orthogonality and 0

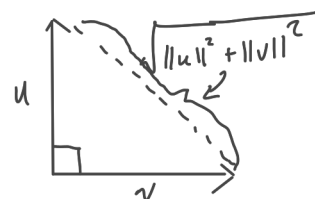
- (a) 0 is orthogonal to every vector in V .
 (b) 0 is the only vector in V that is orthogonal to itself.

Proof (a) $\langle 0, v \rangle = 0$ for all $v \in V$ by a previous result
 (b) $\langle v, v \rangle = 0$ if & only if $v = 0$ by the definiteness of the inner product.

6.12 Pythagorean theorem

Suppose $u, v \in V$. If u and v are orthogonal, then

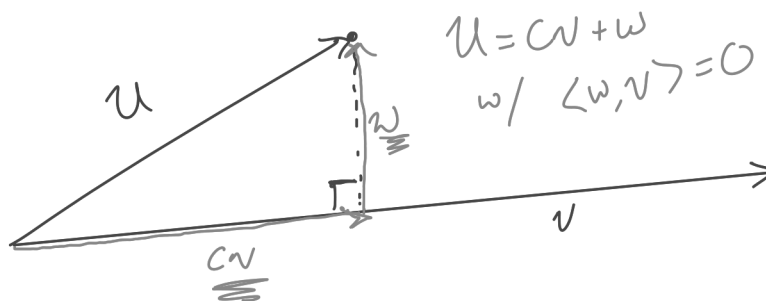
$$\|u + v\|^2 = \|u\|^2 + \|v\|^2.$$



Proof § $\langle u, v \rangle = 0$. Then,

$$\begin{aligned} \|u + v\|^2 &= \langle u + v, u + v \rangle \\ &= \langle u, u + v \rangle + \langle v, u + v \rangle \\ &= \langle u, u \rangle + \langle u, v \rangle + \langle v, u \rangle + \langle v, v \rangle \\ &= \|u\|^2 + \langle u, v \rangle + \overline{\langle u, v \rangle} + \|v\|^2 \\ &= \|u\|^2 + 0 + 0 + \|v\|^2 \\ &= \|u\|^2 + \|v\|^2 \end{aligned}$$

Idea



6.13 an orthogonal decomposition

Suppose $u, v \in V$, with $v \neq 0$. Set $c = \frac{\langle u, v \rangle}{\|v\|^2}$ and $w = u - \frac{\langle u, v \rangle}{\|v\|^2}v$. Then

$$u = cv + w \quad \text{and} \quad \langle w, v \rangle = 0.$$

proof

$$cv + w = \frac{\langle u, v \rangle}{\|v\|^2} v + u - \frac{\langle u, v \rangle}{\|v\|^2} v = u$$

$$\begin{aligned} \langle w, v \rangle &= \left\langle u - \frac{\langle u, v \rangle}{\|v\|^2} v, v \right\rangle = \langle u, v \rangle - \left\langle \frac{\langle u, v \rangle}{\|v\|^2} v, v \right\rangle \\ &= \langle u, v \rangle - \frac{\langle u, v \rangle}{\|v\|^2} \langle v, v \rangle \\ &= \langle u, v \rangle - \langle u, v \rangle \frac{\|v\|^2}{\|v\|^2} = 0 \end{aligned}$$

6.14 Cauchy-Schwarz inequality (Cauchy-Bunyakovsky-Schwarz inequality)

Suppose $u, v \in V$. Then

$$|\langle u, v \rangle| \leq \|u\| \|v\|.$$

This inequality is an equality if and only if one of u, v is a scalar multiple of the other.

proof

(assume $v \neq 0$)

If $u=0$, then both sides of the inequality are zero, so it holds. If not, then $u = cv + w$ where $\langle w, v \rangle = 0$. Using the pythagorean theorem, we get

$$\begin{aligned} \|u\|^2 &= \|cv + w\|^2 = \|cv\|^2 + \|w\|^2 \quad (\text{since } \langle cv, w \rangle = c \langle v, w \rangle = 0) \\ &= \left\| \frac{\langle u, v \rangle}{\|v\|^2} v \right\|^2 + \|w\|^2 \\ &= \frac{|\langle u, v \rangle|^2}{\|v\|^4} \|v\|^2 + \|w\|^2 \\ &= \frac{|\langle u, v \rangle|^2}{\|v\|^2} + \|w\|^2 \\ &\geq \frac{|\langle u, v \rangle|^2}{\|v\|^2} \end{aligned}$$

equality if $w=0$
 $\nexists u=cv$

$$\text{So, } \|u\|^2 \geq \frac{|\langle u, v \rangle|^2}{\|v\|^2} \quad \nexists \quad \|u\|^2 \|v\|^2 \geq |\langle u, v \rangle|^2, \text{ or } |\langle u, v \rangle| \leq \|u\| \|v\|.$$

6.17 triangle inequality

Suppose $u, v \in V$. Then

$$\|u + v\| \leq \|u\| + \|v\|.$$

This inequality is an equality if and only if one of u, v is a nonnegative real multiple of the other.

Proof

$$\begin{aligned} \|u+v\|^2 &= \langle u+v, u+v \rangle \\ &= \langle u, u \rangle + \langle u, v \rangle + \langle v, u \rangle + \langle v, v \rangle \\ &= \|u\|^2 + \|v\|^2 + \langle u, v \rangle + \overline{\langle u, v \rangle} \\ &= \|u\|^2 + \|v\|^2 + 2 \operatorname{Re}(\langle u, v \rangle) \\ &\leq \|u\|^2 + \|v\|^2 + 2 |\langle u, v \rangle| \\ &\leq \|u\|^2 + \|v\|^2 + 2 \|u\| \|v\| \quad (\text{via CS}) \\ &= (\|u\| + \|v\|)^2 \end{aligned}$$

$$\begin{aligned} z &= x + iy \\ \bar{z} &= x - iy \\ z + \bar{z} &= 2x \\ \text{i.e. } z + \bar{z} &= 2 \operatorname{Re}(z) \\ |z| &= \sqrt{x^2 + y^2} \\ &\geq x \\ \text{So, } 2|z| &\geq 2 \operatorname{Re}(z) \geq 2 \operatorname{Re}(z) \end{aligned}$$

$$\text{So, } \|u+v\|^2 \leq (\|u\| + \|v\|)^2 \quad \& \quad \|u+v\| \leq \|u\| + \|v\|$$

6.21 parallelogram equality

Suppose $u, v \in V$. Then

$$\|u+v\|^2 + \|u-v\|^2 = 2(\|u\|^2 + \|v\|^2).$$

Proof

$$\begin{aligned} \|u+v\|^2 + \|u-v\|^2 &= \langle u+v, u+v \rangle + \langle u-v, u-v \rangle \\ &= \langle u, u \rangle + \langle u, v \rangle + \langle v, u \rangle + \langle v, v \rangle \\ &\quad + \langle u, u \rangle - \langle u, v \rangle - \langle v, u \rangle + \langle v, v \rangle \\ &= 2\langle u, u \rangle + 2\langle v, v \rangle \\ &= 2(\|u\|^2 + \|v\|^2) \end{aligned}$$

