

5E Commuting Operators

i.e. $S, T \in \mathcal{L}(V)$

5.71 definition: *commute*

- Two operators S and T on the same vector space commute if $\underline{ST} = \underline{TS}$.
- Two square matrices $\underline{A}$ and $\underline{B}$ of the same size commute if $\underline{AB} = \underline{BA}$.

5.74 commuting operators correspond to commuting matrices

Suppose $S, T \in \mathcal{L}(V)$ and $v_1, \dots, v_n$ is a basis of V . Then $\underline{S}$ and $\underline{T}$ commute if and only if $\underline{M}(S, (v_1, \dots, v_n))$ and $\underline{M}(T, (v_1, \dots, v_n))$ commute.

Proof

$$\begin{aligned} S \text{ \& } T \text{ commute} &\Leftrightarrow ST = TS \\ &\Leftrightarrow \underline{M}(ST) = \underline{M}(TS) \\ &\Leftrightarrow \underline{M}(S)\underline{M}(T) = \underline{M}(T)\underline{M}(S) \\ &\Leftrightarrow \underline{M}(S) \text{ \& } \underline{M}(T) \text{ commute} \end{aligned}$$

5.75 eigenspace is invariant under commuting operator

Suppose $\underline{S}, \underline{T} \in \mathcal{L}(V)$ commute and $\lambda \in \mathbf{F}$. Then $\underline{E}(\lambda, \underline{S})$ is invariant under $\underline{T}$.

Proof

We want to show that $v \in E(\lambda, S)$ implies $Tv \in E(\lambda, S)$.

$\S v \in E(\lambda, S)$. Then,

$$S(Tv) = TSv = T(Sv) = T(\lambda v) = \lambda(Tv) \in E(\lambda, S)$$

i.e. $STv = \lambda Tv$, so $(S - \lambda I)(Tv) = 0 \text{ \& } Tv \in E(\lambda, S)$.

Notice: $E(\lambda, T)$ is also invariant under S (just swap $S \text{ \& } T$ in this proof).

5.76 simultaneous diagonalizability $\Leftrightarrow$ commutativity

Two diagonalizable operators on the same vector space have diagonal matrices with respect to the same basis if and only if the two operators commute.

Proof $\S$ $S, T \in \mathcal{L}(V)$ have diagonal matrices wrt. a basis $v_1, \dots, v_n$ of V . Then, $M(S)M(T) = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \begin{bmatrix} \eta_1 & & 0 \\ & \ddots & \\ 0 & & \eta_n \end{bmatrix}$
 $= \begin{bmatrix} \lambda_1 \eta_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \eta_n \end{bmatrix}$
 $= \begin{bmatrix} \eta_1 \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \eta_n \lambda_n \end{bmatrix}$
 $= \begin{bmatrix} \eta_1 & & 0 \\ & \ddots & \\ 0 & & \eta_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} = M(T)M(S)$ So, S & T commute.

In the other direction, $\S$ $S, T \in \mathcal{L}(V)$ are diagonalizable commuting operators.

Let $\lambda_1, \dots, \lambda_m$ denote the distinct eigenvalues of S . Then, we have

$$V = E(\lambda_1, S) \oplus \dots \oplus E(\lambda_m, S) \quad (\text{since } S \text{ is diagonalizable})$$

Since $E(\lambda_k, S)$ is invariant under T for $k \in \{1, \dots, m\}$, $T|_{E(\lambda_k, S)} \in \mathcal{L}(E(\lambda_k, S))$ & is also diagonalizable (since T is). So, $E(\lambda_k, S)$ has a basis of eigenvectors of $T|_{E(\lambda_k, S)}$ & hence of T . Combining bases for each, we get a basis of V that consists of eigenvectors of T which are also eigenvectors of S . Hence, the matrices of S & T wrt. this basis are both diagonal.

5.78 common eigenvector for commuting operators

Every pair of commuting operators on a finite-dimensional nonzero complex vector space has a common eigenvector.

Proof $\S$ V is a nonzero complex finite-dim. V.S.

$\S$ That $S, T \in \mathcal{L}(V)$ commute. Let λ be an eigenvalue of S (which exists since V is complex & finite-dim.). So, $E(\lambda, S) \neq \{0\}$ is invariant under T & therefore $T|_{E(\lambda, S)} \in \mathcal{L}(E(\lambda, S))$.

So, $T|_{E(\lambda, S)}$ has an eigenvalue (for the same reason S does) & must therefore have an eigenvector. But, this eigenvector is in $E(\lambda, S)$, so it is also an eigenvector of S .

5.80 commuting operators are simultaneously upper triangularizable

Suppose V is a finite-dimensional complex vector space and S, T are commuting operators on V . Then there is a basis of V with respect to which both S and T have upper-triangular matrices.

Proof We do induction on $n = \dim V$. ^{Base case:} For $n=1$, every matrix of any operator is diagonal, so we are done.

Inductive Step: $\S$ the result is true for all values less than $n = \dim V$. Let $S, T \in \mathcal{L}(V)$ be commuting operators. Then, by our previous result, $S \S T$ have a common eigenvector. Let v_1 be such an eigenvector of both $S \S T$. Then, $\underline{Sv_1 \in \text{span}(v_1)} \S \underline{Tv_1 \in \text{span}(v_1)}$.

So, let $W \subseteq V$ s.t. $V = \text{span}(v_1) \oplus W$. Define $P \in \mathcal{L}(V, V)$ by $P(v) = P(av_1 + w) = w$. Then, we will show $PS|_W, PT|_W \in \mathcal{L}(W)$ (here, $S|_W \in \mathcal{L}(W, V) \S T|_W \in \mathcal{L}(W, V)$) commute.

$$\begin{aligned} ((PS|_W)(PT|_W)w) &= (PS|_W)(\underbrace{Tw}_{= \tilde{w} + av_1}) = P(STw - Sav_1) = P(STw - \tilde{a}v_1) = PSTw = PTSw = (PT|_W)(PS|_W)w \end{aligned}$$

where
 $Tw = \tilde{w} + av_1$
 for $\tilde{w} \in W$

So, $PS|_W \S PT|_W$ are commuting operators on W $\S \dim W = \dim V - 1 = n - 1 < n$. Therefore, the inductive hypothesis applies $\S$ both can be simultaneously upper-triangularized. Hence, taking $v_2, \dots, v_n$ to be the basis of W doing this, $v_1, \dots, v_n$ simultaneously upper-triangularizes $S \S T$, since $Sv_k \in \text{span}(v_1, \dots, v_k) \S Tv_k \in \text{span}(v_1, \dots, v_k)$.

5.81 eigenvalues of sum and product of commuting operators

Suppose V is a finite-dimensional complex vector space and S, T are commuting operators on V . Then

- every eigenvalue of $S + T$ is an eigenvalue of S plus an eigenvalue of T ,
- every eigenvalue of ST is an eigenvalue of S times an eigenvalue of T .

Proof First, note that if A, B are upper-triangular matrices, then so is $A+B$ & AB . Moreover, the diagonals of $A+B$ and AB are the sums (resp. products) of the diagonal entries on A & B . That is,

$$A+B = \begin{bmatrix} a_{1,1} & \dots & a_{1,n} \\ & \ddots & \\ 0 & & a_{n,n} \end{bmatrix} + \begin{bmatrix} b_{1,1} & \dots & b_{1,n} \\ & \ddots & \\ 0 & & b_{n,n} \end{bmatrix} = \begin{bmatrix} (a_{1,1}+b_{1,1}) & \dots & (a_{1,n}+b_{1,n}) \\ & \ddots & \\ 0 & & (a_{n,n}+b_{n,n}) \end{bmatrix}$$

$$AB = \begin{bmatrix} a_{1,1} & \dots & a_{1,n} \\ & \ddots & \\ 0 & & a_{n,n} \end{bmatrix} \begin{bmatrix} b_{1,1} & \dots & b_{1,n} \\ & \ddots & \\ 0 & & b_{n,n} \end{bmatrix} = \begin{bmatrix} (a_{1,1}b_{1,1}) & & \\ & \ddots & \\ 0 & & (a_{n,n}b_{n,n}) \end{bmatrix}$$

So, since there exists a basis of V w.r.t. which both S & T have upper triangular matrices, and since the entries on the diagonal of an UT matrix of an operator are the eigenvalues of the operator, we are done!