5.62 *necessary and sufficient condition for diagonalizability*

Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Then T is diagonalizable if and only if the minimal polynomial of T equals $(z-\lambda_1)\cdots(z-\lambda_m)$ for some list of distinct numbers $\lambda_1, \ldots, \lambda_m \in \mathbf{F}$.

Proof First, suppose T is diagonalizable. Then, V has a basis of e-vec's of T. Let $v_1, \ldots, v_n$ be such a basis & let $\lambda_1, \ldots, \lambda_m$ be the distinct eigenvalues of T. Then, for each v_k, there exist λ_j s.t. $(T-\lambda_j I)v_k = 0$. So, $(T-\lambda_1 I)\cdots(T-\lambda_m I) = 0$ since it is zero on a basis (as $p(T)q(T) = q(T)p(T)$ for polys p, q). Hence, the min. poly. of T is a poly. multiple of $(z-\lambda_1)\cdots(z-\lambda_m)$ & is in fact the minimal poly of T, since any divisor of it leaves some eigenspace out of its null space.

Now, in the other direction, suppose the min. poly. of T is $(z-\lambda_1)\cdots(z-\lambda_m)$ for some list of distinct $\lambda_1, \ldots, \lambda_m \in \mathbb{F}$.
We will do induction on m.
Base Case: Let $m=1$. Then, $T-\lambda_1 I = 0$ & $T = \lambda_1 I$ so T is a scalar multiple of the identity & $\mathcal{M}(T)$ is diagonal w.r.t. any basis of V (convince yourself of this by recalling the def'n of $\mathcal{M}(T)$).
Induction Step: Suppose our desired result is true for all values less than $m>1$, and let the min. poly. of T be $(z-\lambda_1)\cdots(z-\lambda_m)$ for some list of distinct $\lambda_1, \ldots, \lambda_m \in \mathbb{F}$. We will show that range$(T-\lambda_m I)$ has a basis of e-vectors of T & that $V = \text{range}(T-\lambda_m I) \oplus \text{null}(T-\lambda_m I)$, showing then that any extension of this basis of e-vectors for range$(T-\lambda_m I)$ adds only more eigenvectors of T, and hence that V has a basis of e-vec's of T & is therefore diagonalizable.

$\boxed{\text{Proof}}$ (continued)

First, note that $\text{range}(T-\lambda_m I)$ is an invariant subspace of T (since $\text{range}\, p(T)$ is invariant under T for any poly p). So, $T|_{\text{range}(T-\lambda_m I)} \in \mathcal{L}(\text{range}(T-\lambda_m I))$. Now, if $u \in \text{range}(T-\lambda_m I)$, then $\exists v \in V$ s.t. $(T-\lambda_m I)v = u$. So,

$$\underline{(T-\lambda_1 I)\cdots(T-\lambda_{m-1} I)u} = \underline{(T-\lambda_1 I)\cdots(T-\lambda_{m-1} I)(T-\lambda_m I)v} = 0.$$

Hence, $(z-\lambda_1)\cdots(z-\lambda_{m-1})$ is a poly. mult. of the min. poly. of $T|_{\text{range}(T-\lambda_m I)}$. By our inductive hypothesis, this means that $T|_{\text{range}(T-\lambda_m I)}$ is diagonalizable & $\underline{\text{range}(T-\lambda_m I)}$ has $\underline{\text{a basis}}$ of $\underline{\text{e-vec}}$'s of $T|_{\text{range}(T-\lambda_m I)}$ (and therefore also $\underline{\text{of } T}$).

Now, to show $V = \text{range}(T-\lambda_m I) \oplus \text{null}(T-\lambda_m I)$, let $u \in \text{range}(T-\lambda_m I) \cap \text{null}(T-\lambda_m I)$. we show $u=0$ to get $\text{range}(T-\lambda_m I) \cap \text{null}(T-\lambda_m I) = \{0\}$ & get $V = \underline{\text{range}(T-\lambda_m I)} \oplus \underline{\text{null}(T-\lambda_m I)}$. Since $u \in \text{null}(T-\lambda_m I)$, $Tu = \lambda_m u$. By our argument before,

$$0 = (T-\lambda_1 I)\cdots(T-\lambda_{m-1} I)u = (\lambda_m-\lambda_1)\cdots(\lambda_m-\lambda_{m-1})u$$

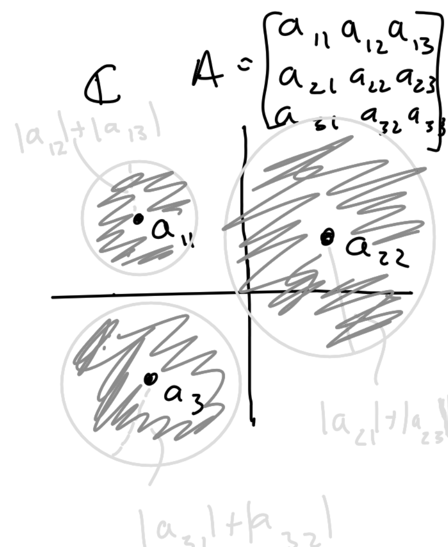
Since $\lambda_m - \lambda_k \neq 0$ for $k \neq m$ (as $\lambda_1, \ldots, \lambda_m$ are distinct), this means $u=0$. So, we are done!

$\underline{\text{(Wow, what a proof!)}}$

5.65 *restriction of diagonalizable operator to invariant subspace*

Suppose $T \in \mathcal{L}(V)$ is diagonalizable and U is a subspace of V that is invariant under T. Then $T|_U$ is a diagonalizable operator on U.

$\boxed{\text{Proof}}$ If T is diagonalizable, then the min. poly. of T is of the form $(z-\lambda_1)\cdots(z-\lambda_m)$ for some list of distinct $\lambda_1, \dots, \lambda_m \in \mathbb{F}$. So, $T|_U$ has a min. poly. of this form as well, since its min. poly. divides the min. poly. of T. Hence, it is also diagonalizable.



5.66 definition: *Gershgorin disks*

Suppose $T \in \mathcal{L}(V)$ and $v_1, \dots, v_n$ is a basis of V. Let A denote the matrix of T with respect to this basis. A *Gershgorin disk* of T with respect to the basis $v_1, \dots, v_n$ is a set of the form

Here, $\mathbb{F} = \mathbb{C}$ or $\mathbb{F} = \mathbb{R}$

$$\left\{ z \in \mathbf{F} : |z - A_{j,j}| \le \sum_{\substack{k=1 \\ k \neq j}}^{n} |A_{j,k}| \right\},$$

where $j \in \{1, \dots, n\}$.

5.67 *Gershgorin disk theorem*

Suppose $T \in \mathcal{L}(V)$ and $v_1, \dots, v_n$ is a basis of V. Then each eigenvalue of T is contained in some Gershgorin disk of T with respect to the basis $v_1, \dots, v_n$.

→ All the e-vals live in the green part

$\boxed{\text{Proof}}$ Let $\lambda \in \mathbb{F}$ be an eigenvalue of T. Let $w \in V$ be a corresponding e-vec. Since $v_1, \dots, v_n$ is a basis of V, $\exists\, c_1, \dots, c_n \in \mathbb{F}$ s.t. $w = c_1 v_1 + \dots + c_n v_n$. Now, apply T to both sides, writing $A = \mathcal{M}(T)$ wrt. the basis $v_1, \dots, v_n$:

$$\underline{\lambda w} = c_1 T v_1 + \dots + c_n T v_n = \sum_{k=1}^{n} c_k \sum_{j=1}^{n} A_{j,k} v_j = \sum_{k=1}^{n} \sum_{j=1}^{n} c_k A_{jk} v_j = \sum_{j=1}^{n} \left(\sum_{k=1}^{n} c_k A_{jk} \right) v_j.$$

so, $\sum_{k=1}^{n} c_k A_{jk} = \lambda c_j$. Hence, $\lambda c_j - A_{jj} c_j = \sum_{\substack{k=1 \\ k \neq j}}^{n} c_k A_{jk}$ & $|\lambda - A_{jj}| \le \sum_{\substack{k=1 \\ k \neq j}}^{n} \frac{|c_k|}{|c_j|} |A_{jk}|$.

so, if $|c_j|$ is the max of $|c_1|, \dots, |c_n|$, $\frac{|c_k|}{|c_j|} \le 1$ & we're done!