5D Diagonalizable Operators

5.48 definition: *diagonal matrix*

A *diagonal matrix* is a square matrix that is 0 everywhere except possibly on the diagonal.

$$\begin{bmatrix} * & & 0 \\ & a_{ii} & \\ 0 & & * \end{bmatrix}\begin{bmatrix}0\\ \vdots\\ 0\\ 1\\ 0\\ \vdots\\ 0\end{bmatrix} = a_{ii}\begin{bmatrix}0\\ \vdots\\ 0\\ 1\\ 0\\ \vdots\\ 0\end{bmatrix}$$

5.50 definition: *diagonalizable*

An operator on V is called *diagonalizable* if the operator has a diagonal matrix with respect to some basis of V.

Notice! If $T \in \mathcal{L}(V)$ is diagonalizable, then $Tv_k = \mathcal{M}(T)_{kk} v_k$ for all v_k in the basis $v_1, \ldots, v_n$ diagonalizing $\mathcal{M}(T)$

5.52 definition: *eigenspace*, $E(\lambda, T)$

Suppose $T \in \mathcal{L}(V)$ and $\lambda \in \mathbf{F}$. The *eigenspace* of T corresponding to λ is the subspace $E(\lambda, T)$ of V defined by

$$E(\lambda, T) = \operatorname{null}(T - \lambda I) = \{v \in V : Tv = \lambda v\}.$$

Hence $E(\lambda, T)$ is the set of all eigenvectors of T corresponding to λ, along with the 0 vector. (i.e. $E(\lambda, T) \neq \{0\} \iff \lambda$ is an eigenvalue of T)

5.54 *sum of eigenspaces is a direct sum*

Suppose $T \in \mathcal{L}(V)$ and $\lambda_1, \ldots, \lambda_m$ are distinct eigenvalues of T. Then

$$E(\lambda_1, T) + \cdots + E(\lambda_m, T)$$

is a direct sum. Furthermore, if V is finite-dimensional, then

$$\dim E(\lambda_1, T) + \cdots + \dim E(\lambda_m, T) \leq \dim V.$$

Good!

$\boxed{\text{Proof}}$ To show $E(\lambda_1, T) + \cdots + E(\lambda_m, T)$ is direct, suppose $v_1 + \cdots + v_m = 0$ where $v_i \in E(\lambda_i, T)$, $i \in \{1, \ldots, m\}$. Since any nonzero v_i is an eigenvector of T corresponding to the eigenvalue λ_i, this is a sum of zero or more eigenvectors corresponding to distinct eigenvalues of T. But, eigenvectors corresponding to distinct eigenvalues are lin. indep., so this sum must have zero nonzero terms (i.e. $v_i = 0$ for $i \in \{1, \ldots, m\}$). Hence, the sum is direct. Consequently, $\dim V \geq \dim(E(\lambda_1, T) + \cdots + E(\lambda_m, T)) = \dim E(\lambda_1, T) + \cdots + \dim E(\lambda_m, T)$.

5.55 *conditions equivalent to diagonalizability*

Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Let $\lambda_1, \ldots, \lambda_m$ denote the distinct eigenvalues of T. Then the following are equivalent.

(a) T is diagonalizable.

(b) V has a basis consisting of eigenvectors of T.

(c) $V = E(\lambda_1, T) \oplus \cdots \oplus E(\lambda_m, T)$.

(d) $\dim V = \dim E(\lambda_1, T) + \cdots + \dim E(\lambda_m, T)$.

$\boxed{\text{Proof}}$ (a) $\Rightarrow$ (b) Suppose T is diagonalizable. Let $v_1, \ldots, v_n$ be a basis of V s.t. $\mathcal{M}(T) = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$ for some $\lambda_1, \ldots, \lambda_n \in \mathbb{F}$. Then, by the definition of $\mathcal{M}(T)$, $Tv_k = \mathcal{M}(T)_{k,k} v_k = \lambda_k v_k$. So, λ_k is an eigenvalue of T since $v_k \neq 0$ (and v_k is then an eigenvector).

(b) $\Rightarrow$ (c) Suppose V has a basis consisting of e-vec's of T. Then, every $v \in V$ is a linear combo of e-vec's of T. Thus, it is a sum of elements from $E(\lambda_1, T), \ldots, E(\lambda_m, T)$ where $\lambda_1, \ldots, \lambda_m$ are all the distinct eigenvalues of T. So, $V = E(\lambda_1, T) + \cdots + E(\lambda_m, T)$ and by our previous result this sum is direct.

(c) $\Rightarrow$ (d) Suppose $V = E(\lambda_1, T) \oplus \cdots \oplus E(\lambda_m, T)$. Then, we have $\dim V = \dim\left(E(\lambda_1, T) \oplus \cdots \oplus E(\lambda_m, T)\right) = \dim E(\lambda_1, T) + \cdots + \dim E(\lambda_m, T)$.

(d) $\Rightarrow$ (a) Suppose $\dim V = \dim E(\lambda_1, T) + \cdots + \dim E(\lambda_m, T)$. Take a basis for each $E(\lambda_i, T)$ & put them all together as a list $v_1, \ldots, v_n$ w/ $n = \dim V$. Suppose $a_1, \ldots, a_n \in \mathbb{F}$ s.t. $a_1 v_1 + \cdots + a_n v_n = 0$. Write u_k for the portion of the sum belonging to $E(\lambda_k, T)$. So $u_1 + \cdots + u_m = 0$. Since eigenvectors corresponding to distinct e-val's are lin. indep., and since each u_k is an eigenvector if $u_k \neq 0$, this must mean $u_k = 0$ for $k = 1, \ldots, m$. So, all the a_i must be zero as well & $v_1, \ldots, v_n$ is a lin. indep. list of length $\dim V$, & hence a basis of V.

5.58 *enough eigenvalues implies diagonalizability*

Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$ has $\dim V$ distinct eigenvalues. Then T is diagonalizable.

$\boxed{\text{Ex}}$ Let $T \in \mathcal{L}(\mathbb{F}^3)$ defined by $T(x,y,z) = (2x+y, 5y+3z, 8z)$.

Wrt. the basis $(1,0,0)$, $(0,1,0)$, $(0,0,1)$, $\mathcal{M}(T) = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 5 & 3 \\ 0 & 0 & 8 \end{bmatrix}$.

This is not diagonal, but it shows that the eigenvalues of T are $2, 5$, and 8. So, T has $\dim \mathbb{F}^3 = 3$ distinct eigenvalues. Hence, it can be diagonalized. To do this, just find an e-vector corresponding to each e-val here.

One choice: $\lambda_1 = 2 \rightsquigarrow v_1 = (1,0,0)$, $\lambda_2 = 5 \rightsquigarrow v_2 = (1,3,0)$, $\lambda_3 = 8 \rightsquigarrow v_3 = (1,6,6)$.

Then, if $v = a_1 v_1 + a_2 v_2 + a_3 v_3$,
$$\begin{aligned} Tv &= a_1 T v_1 + a_2 T v_2 + a_3 T v_3 \\ &= 2a_1 v_1 + 5a_2 v_2 + 8a_3 v_3 \end{aligned}$$

So, this lets us know what T does to any vector $v \in V$ since v_1, v_2, v_3 is a basis. For instance, this makes computing powers of T easy!

$$T^k v = T^{k-1}(2a_1 v_1 + 5a_2 v_2 + 8a_3 v_3) = \cdots\cdots = 2^k a_1 v_1 + 5^k a_2 v_2 + 8^k a_3 v_3$$