5C Upper-Triangular Matrices

5.35 definition: *matrix of an operator, $\mathcal{M}(T)$*

Suppose $T \in \mathcal{L}(V)$. The *matrix of* T with respect to a basis $v_1, \dots, v_n$ of V is the n-by-n matrix

$$\mathcal{M}(T) = \begin{pmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & & \vdots \\ A_{n,1} & \cdots & A_{n,n} \end{pmatrix}$$

whose entries $A_{j,k}$ are defined by

$$Tv_k = A_{1,k}v_1 + \cdots + A_{n,k}v_n.$$

The notation $\mathcal{M}(T, (v_1, \dots, v_n))$ is used if the basis is not clear from the context.

$\boxed{\text{Recall}}$ If $T \in \mathcal{L}(V, W)$ & $w_1, \dots, w_m$ is a basis of W, $v_1, \dots, v_n$ is a basis of V, Then $\mathcal{M}(T)$ is the matrix s.t.

$$Tv_k = \mathcal{M}(T)_{1,k} w_1 + \dots + \mathcal{M}(T)_{m,k} w_m$$

Same idea (but w/ $W = V$ & the same basis for both)

$\boxed{\text{Ex}}$ Let $T \in \mathcal{L}(\mathbb{R}^3)$ be $T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2x+y \\ 5y+3z \\ 8z \end{bmatrix}$.

Then, w.r.t. the basis $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ of $\mathbb{F}^3$,

$$\mathcal{M}(T) = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 5 & 3 \\ 0 & 0 & 8 \end{bmatrix}$$

5.37 definition: *diagonal of a matrix*

The *diagonal* of a square matrix consists of the entries on the line from the upper left corner to the bottom right corner.

$\boxed{\text{Ex}}$ the diagonal of $\begin{bmatrix} 2 & 1 & 0 \\ 0 & 5 & 3 \\ 0 & 0 & 8 \end{bmatrix}$ is circled here

5.38 definition: *upper-triangular matrix*

A square matrix is called *upper triangular* if all entries below the diagonal are 0.

$\boxed{\text{EX}}$ The matrix $\begin{bmatrix} 2 & 1 & 0 \\ 0 & 5 & 3 \\ 0 & 0 & 8 \end{bmatrix}$ is upper-triangular.

all 0's

The matrix $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ is not.

not all 0's

5.39 *conditions for upper-triangular matrix*

Suppose $T \in \mathcal{L}(V)$ and $v_1, \ldots, v_n$ is a basis of V. Then the following are equivalent.

(a) The matrix of T with respect to $v_1, \ldots, v_n$ is upper triangular.

(b) $\operatorname{span}(v_1, \ldots, v_k)$ is invariant under T for each $k = 1, \ldots, n$.

(c) $Tv_k \in \operatorname{span}(v_1, \ldots, v_k)$ for each $k = 1, \ldots, n$.

$\boxed{\text{Proof}}$ (a)$\Rightarrow$(b) § $\mathcal{M}(T)$ is U.T. w.r.t. the basis $v_1, \ldots, v_n$. Then,

$$Tv_j = \mathcal{M}(T)_{1,j}v_1 + \cdots + \mathcal{M}(T)_{n,j}v_n$$
$$= \mathcal{M}(T)_{1,j}v_1 + \cdots + \mathcal{M}(T)_{j,j}v_j \qquad (\text{since } \mathcal{M}(T)_{i,j} = 0 \text{ for } i > j)$$

So, $Tv_j \in \operatorname{span}(v_1, \ldots, v_j) \subseteq \operatorname{span}(v_1, \ldots, v_k)$ for $k \geq j$. Hence, $\operatorname{span}(v_1, \ldots, v_k)$ is invariant under T for $k = 1, \ldots, n$.

(b)$\Rightarrow$(c) § $\operatorname{span}(v_1, \ldots, v_k)$ is invariant under T for $k = 1, \ldots, n$.

Then, $Tv \in \operatorname{span}(v_1, \ldots, v_k)$ for any $v \in \operatorname{span}(v_1, \ldots, v_k)$. Since $v_k \in \operatorname{span}(v_1, \ldots, v_k)$ we may conclude $Tv_k \in \operatorname{span}(v_1, \ldots, v_k)$ for $k = 1, \ldots, n$.

(c)$\Rightarrow$(a) § $Tv_k \in \operatorname{span}(v_1, \ldots, v_k)$ for $k = 1, \ldots, n$.

Then, $Tv_k = a_1v_1 + \cdots + a_kv_k$ for some $a_1, \ldots, a_k \in \mathbb{F}$.

But this means $Tv_k = a_1v_1 + \cdots + a_nv_n$ w/ $a_i = 0$ for $i > k$.

Since $v_1, \ldots, v_n$ is a basis, this representation (i.e. the coefficients $a_1, \ldots, a_n$) are unique. But this gives $Tv_k = \mathcal{M}(T)_{1,k}v_1 + \cdots + \mathcal{M}(T)_{n,k}v_n$ has $\mathcal{M}(T)_{i,k} = 0$ for $i > k$.

That is, $\mathcal{M}(T)$ is U.T.

5.40 *equation satisfied by operator with upper-triangular matrix*

Suppose $T \in \mathcal{L}(V)$ and V has a basis with respect to which T has an upper-triangular matrix with diagonal entries $\lambda_1, \ldots, \lambda_n$. Then

$$(T - \lambda_1 I) \cdots (T - \lambda_n I) = 0.$$

This means that $P(T) = 0$ where $P(z) = (z-\lambda_1) \cdots (z-\lambda_n)$.

This must be a poly. multiple of the min. poly. of T. So the min. poly. of T factors completely into linear terms. Moreover, some of these λ_i's are eigenvalues of T (in fact, $\lambda_1, \ldots, \lambda_n$ must contain all of T's eigenvalues)

Proof Let $v_1, \ldots, v_n$ be a basis of T s.t. $\mathcal{M}(T)$ is upper triangular w/ diagonal entries $\lambda_1, \ldots, \lambda_n$. Since $\mathcal{M}(T)$ is U.T., the previous result tells us $Tv_k \in \operatorname{span}(v_1, \ldots, v_k)$. So, in particular, $Tv_1 = \mathcal{M}(T)_{1,1} v_1 = \lambda_1 v_1$.

Hence, $(T - \lambda_1 I) v_1 = 0$. Since $p(T)q(T) = q(T)p(T)$ for any poly.'s $p, q \in \mathcal{P}(\mathbb{F})$, we have $(T - \lambda_1 I) \cdots (T - \lambda_n I) v_1 = 0$.

Now, observe that $Tv_2 = \mathcal{M}(T)_{1,2} v_1 + \mathcal{M}(T)_{2,2} v_2 = \mathcal{M}(T)_{1,2} v_1 + \lambda_2 v_2$

So, $(T - \lambda_2 I) v_2 = \mathcal{M}(T)_{1,2} v_1 \in \operatorname{span}(v_1)$. Hence, $(T - \lambda_1 I)(T - \lambda_2 I) v_2 = 0$ & (using the fact that $p(T)q(T) = q(T)p(T)$ again) $(T - \lambda_1 I) \cdots (T - \lambda_n I) v_2 = 0$.

Repeat this process to show $(T - \lambda_1 I) \cdots (T - \lambda_n I) v_k = 0$ for $k = 1, \ldots, n$. This tells us that $(T - \lambda_1 I) \cdots (T - \lambda_n I) = 0$ since it is zero on a basis of V.