

5.29 $q(T) = 0 \iff q$ is a polynomial multiple of the minimal polynomial

Suppose V is finite-dimensional, $T \in \mathcal{L}(V)$, and $q \in \mathcal{P}(F)$. Then $q(T) = 0$ if and only if q is a polynomial multiple of the minimal polynomial of T .

Proof $\S$ p is the minimal polynomial of T .
If $q \in \mathcal{P}(F)$ is st. $q(T) = 0$, then doing division,
there are polynomials $s, r \in \mathcal{P}(F)$ st.

$$q = ps + r$$

w/ degree of r less than the degree of p .

Then, $0 = q(T) = p(T)s(T) + r(T) = 0s(T) + r(T) = r(T)$.

So, $r(T) = 0$ $\&$ $\deg r < \deg p$, so by minimality,
 r is the zero polynomial. That is, $q = ps$.

That is, q is a polynomial multiple of p .

Conversely, $\S$ $q = ps$ for some poly $s \in \mathcal{P}(F)$. Then,
 $q(T) = p(T)s(T) = 0s(T) = 0$.

5.31 minimal polynomial of a restriction operator

Suppose V is finite-dimensional, $T \in \mathcal{L}(V)$, and U is a subspace of V that is invariant under T . Then the minimal polynomial of T is a polynomial multiple of the minimal polynomial of $T|_U$.

avoid this mess
 ~~$T: V \rightarrow V$
 $T|_U: U \rightarrow V$
 $\hat{T}|_U: U \rightarrow U$~~

Proof Let p be the min. poly. of T .

Hence, $p(T) = 0$ $\&$ $p(T)v = 0v = 0$ for all $v \in V$.

So, $p(T)u = 0$ for all $u \in U$. Hence, $p(T|_U) = 0$
(where this is $0 \in \mathcal{L}(U)$). Thus, by the previous result, p is
a poly. multiple of the min. poly. of $T|_U$.

5.32 T not invertible $\Leftrightarrow$ constant term of minimal polynomial of T is 0

Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Then T is not invertible if and only if the constant term of the minimal polynomial of T is 0.

Proof Let p be the min. poly. of T .

T is not invertible $\Leftrightarrow 0$ is an eigenvalue of T
 $\Leftrightarrow 0$ is a zero of p
 $\Leftrightarrow p(0) = 0$
 $\Leftrightarrow$ the constant term of p is zero.

Eigenvalues on Odd-Dimensional Real Vector Spaces

5.33 even-dimensional null space

Suppose $F = \mathbb{R}$ and V is finite-dimensional. Suppose also that $T \in \mathcal{L}(V)$ and $b, c \in \mathbb{R}$ with $b^2 < 4c$. Then $\dim \text{null}(T^2 + bT + cI)$ is an even number.

Proof Recall that the null space of $p(T)$ is invariant under T for any polynomial $p \in \mathcal{P}(\mathbb{R})$. If necessary, replace V w/ $\text{null}(T^2 + bT + cI)$ & T with $T|_{\text{null}(T^2 + bT + cI)}$ so that we are only dealing w/ $\text{null}(T^2 + bT + cI)$. This lets us assume $T^2 + bT + cI = 0$ (since we are taking V to be $\text{null}(T^2 + bT + cI)$ if it is not already). We want to show $\dim(\text{null}(T^2 + bT + cI)) = \dim V$ is even. $\S \lambda \in \mathbb{R}$ & $v \in V$ is st. $Tv = \lambda v$. Then,
 $0 = (T^2 + bT + cI)v = \lambda^2 v + b\lambda v + cv = \left((\lambda + \frac{b}{2})^2 + c - \frac{b^2}{4} \right) v$
Since $b^2 < 4c$, $0 < 4c - b^2$ & $0 < c - \frac{b^2}{4}$. So, λ is a positive #.
Hence, $v = 0$. So, T has no eigenvectors.

Proof (continued)

Let U be a subspace of V that is invariant under T . Moreover, $\dim U$ is as large as possible while remaining even. If $U=V$, we are done. If not, let $w \in V$ be some vector st. $w \notin U$. Let $W = \text{span}(w, Tw)$. Then, W is invariant under T , since $T(Tw) = T^2w = -bTw - cw$ (via $T^2 + bT + cI = 0$, which means $T^2 = -bT - cI$). Moreover, since T has no eigenvectors, $Tw \neq \lambda w$ for any $\lambda \in F$ & $\dim W = 2$. Now, $\dim(U+W) = \dim U + \dim W - \dim(U \cap W) = \dim U + 2 - 0$, and $U+W$ is an even-dimensional subspace that is invariant under T , contradicting the maximality of $\dim U$. So, $U=V$ must have been the case!

Every operator on an odd-dimensional vector space has an eigenvalue.

Proof We do induction.

Base case: $\S \dim V = 1$. Then, if $T \in \mathcal{L}(V)$, every nonzero vector $v \in V$ is an eigenvector of T , since $Tv = \lambda v$ for some $\lambda \in \mathbb{R}$ (i.e. if $Tv = v$, then $Tv = (\frac{w}{v})v$).

Induction step: $\S$ any operator on an odd dimensional vector space w/ $\dim V \leq n$ has an eigenvalue. We show this implies any operator on an odd dimensional vector space of dimension $\dim V = n+2$. Let p be the minimal poly. of $T \in \mathcal{L}(V)$. If p is divisible by $(x-\lambda)$, then $p(\lambda) = 0$ & λ is an eigenvalue of T . So, if p is not divisible by $(x-\lambda)$ for any λ , then $p(x) = q(x)(x^2 + bx + c)$ w/ $b^2 < 4c$ (since $x^2 + bx + c = 0$ for $x = \frac{-b \pm \sqrt{b^2 - 4c}}{2}$ & $x \notin \mathbb{R}$ when $b^2 < 4c$).

So, $0 = p(T) = q(T)(T^2 + bT + cI)$. Hence, $\underline{q(T)}|_{\text{range}(T^2 + bT + cI)} = 0$.

Since $\deg q < \deg p$ & p is the min. poly. of T , $\text{range}(T^2 + bT + cI) \neq V$.

By the FTOLM, $\dim V = \dim \text{null}(T^2 + bT + cI) + \dim \text{range}(T^2 + bT + cI)$.

Since $\dim V$ is odd & $\dim \text{null}(T^2 + bT + cI)$ is even, $\dim \text{range}(T^2 + bT + cI)$ is odd. So, our inductive hypothesis applies to $T|_{\text{range}(T^2 + bT + cI)}$.

Hence, $T|_{\text{range}(T^2 + bT + cI)}$ has an eigenvalue, and so then does T .