5B The Minimal Polynomial

5.19 *existence of eigenvalues*

Every operator on a finite-dimensional nonzero complex vector space has an eigenvalue.

$\boxed{\text{Proof}}$ & V is a finite-dim. vector space over $\mathbb{C}$ & $T \in \mathcal{L}(V)$. Let $\dim V = n < \infty$. Choose any $0 \neq v \in V$. Then, the list $v, Tv, T^2v, \ldots, T^nv$ is linearly dependent, since this is a list of length $n+1 > n$. This says there exists coefficients $a_0, a_1, \ldots, a_n \in \mathbb{C}$ s.t. not all are zero &

$$0 = a_0 v + a_1 Tv + a_2 T^2 v + \cdots + a_n T^n v$$

Now, this gives at least one polynomial $\left(\text{ex. } p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n\right)$ s.t. $p(T)v = 0$. Let us suppose that this polynomial is chosen to be of minimal degree. So, by the fundamental theorem of algebra, there exists a poly $q \in \mathcal{P}(\mathbb{C})$ & a value $\lambda \in \mathbb{C}$ s.t.

$$p(z) = (z - \lambda) q(z)$$

So, $0 = p(T)v = (T - \lambda I) q(T) v$. Since q has degree less than p, we know $q(T)v \neq 0$ (Since we supposed).

Hence, λ is an eigenvalue of T & $q(T)v$ is a corresponding eigenvector.

Eigenvalues and the Minimal Polynomial

5.21 definition: *monic polynomial*

A *monic polynomial* is a polynomial whose highest-degree coefficient equals 1.

$\boxed{\text{Ex}}$ $p(z) = 4z + 9z^2$ is not monic
$q(z) = 5z + 11z^2 + z^7$ is monic

5.22 *existence, uniqueness, and degree of minimal polynomial*

Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Then there is a unique monic polynomial $p \in \mathcal{P}(\mathbf{F})$ of smallest degree such that $p(T) = 0$. Furthermore, $\deg p \leq \dim V$.

Not proven yet!

$\boxed{\text{Proof}}$ Suppose $\dim V = n < \infty$ & $T \in \mathcal{L}(V)$.

Write $\text{poly}(T) = \{p(T) : p \in \mathcal{P}(\mathbb{F})\}$. Now, since $0 \in \text{poly}(T)$ (i.e. $0 = 0(T)$) & $q(T) + p(T) = (q+p)(T)$ & $\lambda(q(T)) = (\lambda q)(T)$, $\text{Poly}(T) \leq \mathcal{L}(V)$. Since $\dim V < \infty$, $\dim \mathcal{L}(V) < \infty$ & we can see that $\dim \text{Poly}(T) = m < \infty$ as well. Hence, the list

$$I = T^0, T, T^2, \ldots, T^m$$

is linearly dependent since it is of length $m+1$. Therefore, there exist $\underbrace{a_0, \ldots, a_m}_{\text{not all } 0} \in \mathbb{F}$ s.t. $a_0 I + a_1 T + \cdots + a_m T^m = 0$. & that $K \leq m$ is the largest value s.t. $a_K \neq 0$ (i.e. $a_i = 0$ for $i > K$). Then, the polynomial $\frac{1}{a_K}(a_0 + a_1 z + \cdots + a_K z^K) = p(z)$ is monic and satisfies $p(T) = \frac{1}{a_K}(a_0 I + a_1 T + \cdots + a_K T^K + \cdots + a_m T^m) = \frac{1}{a_K} 0 = 0$. So, such a poly exists. Hence, we can choose one of lowest degree.

Now, to show uniqueness, Suppose p & q are monic polys of minimal degree s.t. $p(T) = q(T) = 0$. Now, $(p-q)(T) = 0$ and notice that if $p - q \neq 0$, then we have a poly of lower degree (since the coefficient on the highest degree term of both p & q is 1) which can be made monic by dividing by the coefficient of the highest power of z in it. this would then contradict minimality, so $p = q$.

5.24 definition: *minimal polynomial*

Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Then the *minimal polynomial* of T is the unique monic polynomial $p \in \mathcal{P}(\mathbf{F})$ of smallest degree such that $p(T) = 0$.

5.27 *eigenvalues are the zeros of the minimal polynomial*

Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$.

(a) The zeros of the minimal polynomial of T are the eigenvalues of T.

(b) If V is a complex vector space, then the minimal polynomial of T has the form

$$(z - \lambda_1)\cdots(z - \lambda_m),$$

where $\lambda_1, \ldots, \lambda_m$ is a list of all eigenvalues of T, possibly with repetitions.

Proof. Let p be the minimal poly. of $T \in \mathcal{L}(V)$.
Now, s'pose λ is a zero of p. Then, by the FToA, $p(z) = (z-\lambda)q(z)$ where q is monic & of degree less than p. Hence, since $p(T)=0$, $(T-\lambda I)q(T)=0$ & b/c q is of lower degree than p, $q(T) \neq 0$. Hence, applying any $v \in V$ s.t. $q(T)v \neq 0$ (means this is possible), we get that $q(T)v \in \text{null}(T-\lambda I)$ & λ is an e.-val. of T.

Now, s'pose λ is an eigenvalue of T. Then, $\exists\, 0 \neq v \in V$ st $Tv = \lambda v$. Accordingly, $T^k v = T^{k-1}\lambda v = \cdots = \lambda^k v$. Hence, $p(T)v = p(\lambda)v$. Since p is the min. poly. of T, $p(T)=0$, so $p(\lambda)v = 0$ & $p(\lambda)=0$ since $v \neq 0$. Thus, λ is a zero of p.

For part b, just use part (a) & the FToA to factor p.