$x^2+1=0$
can't factor over $\mathbb{R}$

5A Invariant Subspaces

5.1 definition: *operator*

A linear map from a vector space to itself is called an *operator*.

Recall that we defined the notation $\mathcal{L}(V)$ to mean $\mathcal{L}(V,V)$.

i.e. operators are maps $T \in \mathcal{L}(V)$

5.2 definition: *invariant subspace*

Suppose $T \in \mathcal{L}(V)$. A subspace U of V is called *invariant* under T if $Tu \in U$ for every $u \in U$.

i.e. U is invariant under T if $T|_U$ is also an operator (i.e. $T|_U \in \mathcal{L}(U)$).

5.5 definition: *eigenvalue*

Suppose $T \in \mathcal{L}(V)$. A number $\lambda \in \mathbf{F}$ is called an *eigenvalue* of T if there exists $v \in V$ such that $v \neq 0$ and $Tv = \lambda v$.

$\boxed{\text{Idea}}$ The smallest nontrivial subspaces of a vector space V are $U = \text{Span}(v)$ for $v \in V$ w/ $v \neq 0$.

So, if we can "break down" V into its smallest components, i.e. $V = V_1 \oplus \cdots \oplus V_m$ w/ $V_i = \text{span}(v_i)$ (w/ $v_i \neq 0$)

Then, if each subspace is invariant under T, we have a good description of what T does: Since every $v \in V$ can be written uniquely as $v = u_1 + \cdots + u_m$ w/ $u_i \in \text{Span}(v_i)$, then $u_i = a_i v_i$ for some $a_i \in \mathbb{F}$. By linearity, $Tv = \underbrace{a_1 Tv_1 + \cdots + a_m Tv_m}$.

So, if V_i is invariant under T, this says $T(av_i) \in V_i$, i.e. $T(av_i) = bv_i$. So, $Tv_i = \overset{\lambda_i}{\frac{b}{a}} v_i$ if $a \neq 0$. Hence, T is just like a 1-dimensional linear map on V_i & for $v \in V$, $Tv = a_1 \lambda_1 v_1 + \cdots + a_m \lambda_m v_m$

That is, we can tell exactly what T does in terms of the most simple linear map by finding 1-dimensional subspaces that are invariant under T.

5.7 *equivalent conditions to be an eigenvalue*

Suppose V is finite-dimensional, $T \in \mathcal{L}(V)$, and $\lambda \in \mathbf{F}$. Then the following are equivalent.

(a) λ is an eigenvalue of T.

(b) $T - \lambda I$ is not injective.

(c) $T - \lambda I$ is not surjective.

(d) $T - \lambda I$ is not invertible.

Reminder: $I \in \mathcal{L}(V)$ is the identity operator. Thus $Iv = v$ for all $v \in V$.

$\boxed{\text{Proof}}$ (sketch) Notice: $Tv = \lambda v$ for some $0 \neq v \in V$ iff only if $Tv - \lambda v = 0$ which says $(T - \lambda I)v = 0$. $\iff$ $T - \lambda I$ has a nontrivial nullspace (i.e. null $T - \lambda I \neq \{0\}$)

$\iff$ $T - \lambda I$ is not injective

$\iff$ $T - \lambda I$ is not surjective

$\iff$ $T - \lambda I$ is not invertible

where the last two $\iff$'s are due to the fact that $T - \lambda I$ is a map between VS.'s of the same dimension.

5.8 definition: *eigenvector*

Suppose $T \in \mathcal{L}(V)$ and $\lambda \in \mathbf{F}$ is an eigenvalue of T. A vector $v \in V$ is called an $\boxed{\textit{eigenvector}}$ of T corresponding to λ if $\boxed{v \neq 0}$ and $\boxed{Tv = \lambda v}$.

5.11 *linearly independent eigenvectors*

Suppose $T \in \mathcal{L}(V)$. Then every list of eigenvectors of T corresponding to distinct eigenvalues of T is linearly independent.

$\boxed{\text{Proof}}$ Suppose there is a linearly dependent list of eigenvectors of $T \in \mathcal{L}(V)$ corresponding to distinct eigenvalues of T. Let $v_1, \dots, v_m$ be this list & suppose further that m is minimal. Necessarily, $m \geq 2$, since a list w/ one nonzero vector is lin. indep. By our assumed linear dependence, $\exists a_1, \dots, a_m \in \mathbb{F}$ s.t. $a_1 v_1 + \dots + a_m v_m = 0$ & not all of the a_i's are zero. By minimality of m, notice that none of the a_i are zero, as if this were the case, we could remove the corresponding v_i from our list. Now, let us apply $T - \lambda_m I$ where λ_m is the eigenvalue corresponding to v_m, to both sides:

$$0 = (T - \lambda_m I)(a_1 v_1 + \dots + a_m v_m) = (T - \lambda_m I)a_1 v_1 + \dots + (T - \lambda_m I)a_m v_m = a_1 T v_1 - \lambda_m a_1 v_1 + \dots + a_{m-1} T v_{m-1} - \lambda_m a_{m-1} v_{m-1}$$
$$= a_1(\lambda_1 - \lambda_m)v_1 + \dots + a_{m-1}(\lambda_{m-1} - \lambda_m)v_{m-1}$$

But notice, since each eigenvalue is distinct, $\lambda_i - \lambda_m \neq 0$ for $i = 1, \dots, m-1$. Hence, we have a linear combo of $v_1, \dots, v_{m-1}$ that is equal to zero & has no coefficient equal to 0. This contradicts the minimality of m!

5.12 *operator cannot have more eigenvalues than dimension of vector space*

Suppose V is finite-dimensional. Then each operator on V has at most $\dim V$ distinct eigenvalues.

Polynomials Applied to Operators

5.13 notation: T^m

Suppose $T \in \mathcal{L}(V)$ and m is a positive integer.

- $T^m \in \mathcal{L}(V)$ is defined by $T^m = \underbrace{T \cdots T}_{m \text{ times}}$.
- T^0 is defined to be the identity operator I on V.
- If T is invertible with inverse T^{-1}, then $T^{-m} \in \mathcal{L}(V)$ is defined by

$$T^{-m} = (T^{-1})^m.$$

5.14 notation: $p(T)$

Suppose $T \in \mathcal{L}(V)$ and $p \in \mathcal{P}(\mathbf{F})$ is a polynomial given by

$$p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_m z^m$$

for all $z \in \mathbf{F}$. Then $p(T)$ is the operator on V defined by

$$p(T) = a_0 I + a_1 T + a_2 T^2 + \cdots + a_m T^m.$$

5.16 definition: *product of polynomials*

If $p, q \in \mathcal{P}(\mathbf{F})$, then $pq \in \mathcal{P}(\mathbf{F})$ is the polynomial defined by

$$(pq)(z) = p(z)q(z)$$

for all $z \in \mathbf{F}$.

5.17 *multiplicative properties*

Suppose $p, q \in \mathcal{P}(\mathbf{F})$ and $T \in \mathcal{L}(V)$. Then

(a) $(pq)(T) = p(T)q(T)$;

(b) $p(T)q(T) = q(T)p(T)$.

Informal proof: When a product of polynomials is expanded using the distributive property, it does not matter whether the symbol is z or T.

$\boxed{\text{Proof}}$ The second part is easy: T commutes w/ itself \$ both P(T) \$ q(T) only involve scaled sums of powers of T.

For a longer answer: (a) $(pq)(z) = p(z)q(z)$ by definition. So, plug in T to get $(pq)(T) = p(T)q(T)$ (this is not entirely correct to say, but is basically the idea). For full clarity, $(pq)(z) = \sum_{j=0}^{m}\sum_{k=0}^{n} a_j b_k z^{j+k}$ where $p(z) = \sum_{j=0}^{m} a_j z^j$ \$ $q(z) = \sum_{k=0}^{n} b_k z^k$. Thus, $(pq)(T) = \sum_{j=0}^{m}\sum_{k=0}^{n} a_j b_k T^{j+k} = \left(\sum_{j=0}^{m} a_j T^j\right)\left(\sum_{k=0}^{n} b_k T^k\right) = p(T)\, q(T)$.

For (b) just use (a) twice: $(pq)(T) = p(T)q(T)$ \$ $(pq)(T) = (qp)(T) = q(T)p(T)$.

5.18 *null space and range of $p(T)$ are invariant under T*

Suppose $T \in \mathcal{L}(V)$ and $p \in \mathcal{P}(\mathbf{F})$. Then null $p(T)$ and range $p(T)$ are invariant under T.

$\boxed{\text{Proof}}$ Suppose $u \in \text{null } p(T)$. Then, $p(T)u = 0$. Thus,

$p(T)(Tu) = T(p(T)u) = T0 = 0$. Hence, $Tu \in \text{null } p(T)$.

Since T is just q(T) for $q(z) = z$

Suppose $v \in \text{range } P(T)$. Then, $\exists\, u \in V$ s.t. $P(T)u = v$.

$Tv = T(P(T)u) = P(T)(Tu)$. So, $Tv \in \text{range } P(T)$.

So, in either case, T does not move things in the null space (resp. range) out of the null space (resp. range) of P(T). Hence, both null P(T) \$ range P(T) are invariant under T.