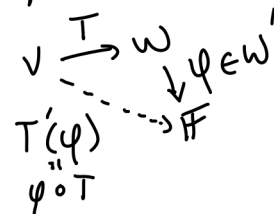


$$U^0 = \{ \varphi \in V' : \varphi(u) = 0 \text{ for all } u \in U \}$$

$$T \in \mathcal{L}(V, W) \\ T' \in \mathcal{L}(W', V')$$



3.125 dimension of the annihilator

Suppose V is finite-dimensional and U is a subspace of V . Then

$$\dim U^0 = \dim V - \dim U.$$

proof Define $i \in \mathcal{L}(U, V)$ by $i(u) = u$. So, $i' \in \mathcal{L}(V', U')$

By the FTOLM, $\dim \text{range } i' + \dim \text{null } i' = \dim V'$. But, notice: $\text{null } i' = \{ \varphi \in V' : i'(\varphi) = 0 \} = \{ \varphi \in V' : \varphi \circ i = 0 \} = \{ \varphi \in V' : \varphi(u) = 0 \forall u \in U \}$. So, $\text{null } i' = U^0$. Since we have shown $\dim V = \dim V'$ when V is finite-dimensional, we have $\dim \text{range } i' + \dim U^0 = \dim V$.

Hence, if $\dim \text{range } i' = \dim U$, we are done!

Notice: $\text{range } i' = \{ \psi \in U' : \exists \varphi \in V' \text{ } i'(\varphi) = \psi \} = \{ \psi \in U' : \exists \varphi \in V' \varphi(u) = \psi(u) \forall u \in U \}$

Since every $\psi \in U'$ can be extended to a linear functional on V (we proved this in the homework), this says $\text{range } i' = U'$ (i.e. i' is surjective).

But, $\dim U' = \dim U$, so we are finished.

3.127 condition for the annihilator to equal $\{0\}$ or the whole space

Suppose V is finite-dimensional and U is a subspace of V . Then

(a) $U^0 = \{0\} \iff U = V$;

(b) $U^0 = V' \iff U = \{0\}$.

proof For part (a): $U^0 = \{0\} \iff \dim U^0 = 0$

$$\iff \dim V - \dim U = 0$$

$$\iff \dim V = \dim U$$

$$\iff V = U$$

For part (b): $U^0 = V' \iff \dim U^0 = \dim V'$

$$\iff \dim V - \dim U = \dim V$$

$$\iff \dim U = 0$$

$$\iff U = \{0\}.$$

3.128 the null space of T'

Suppose V and W are finite-dimensional and $T \in \mathcal{L}(V, W)$. Then

(a) $\text{null } T' = (\text{range } T)^0$; $\checkmark$

(b) $\dim \text{null } T' = \dim \text{null } T + \dim W - \dim V$.

Proof

$\S \psi \in \text{null } T'$. So, $\underline{0} = T'(\psi) = \psi \circ T$. Thus,

for every $v \in V$, $\psi(T(v)) = 0$. Hence, $\psi \in (\text{range } T)^0$. So,
 $\text{null } T' \subseteq (\text{range } T)^0$.

Now, $\S \psi \in (\text{range } T)^0$. Then, $\psi(T(v)) = 0$ for every $v \in V$, i.e.
 $T'(\psi)(v) = 0$ for all $v \in V$. So, $\psi \in \text{null } T'$. Hence, $\text{null } T' \supseteq (\text{range } T)^0$.
 Therefore, $\text{null } T' = (\text{range } T)^0$.

For the second part, observe:

$$\begin{aligned} \dim \text{null } T' &= \dim (\text{range } T)^0 \\ &= \dim W - \dim \text{range } T \\ &= \dim W - (\dim V - \dim \text{null } T) \\ &= \dim \text{null } T + \dim W - \dim V. \end{aligned}$$

Since
 $\dim V = \dim \text{null } T + \dim \text{range } T$

3.129 T surjective is equivalent to T' injective

Suppose V and W are finite-dimensional and $T \in \mathcal{L}(V, W)$. Then

T is surjective $\iff T'$ is injective.

Proof

If $T \in \mathcal{L}(V, W)$ is surjective, then $\text{range } T = W$.

But:

$$\begin{aligned} \text{range } T = W &\iff (\text{range } T)^0 = \{0\} \\ &\iff \text{null } T' = \{0\} \\ &\iff T' \text{ is injective} \end{aligned}$$

So, we are done!

3.130 the range of T'

Suppose V and W are finite-dimensional and $T \in \mathcal{L}(V, W)$. Then

(a) $\dim \text{range } T' = \dim \text{range } T$;

(b) $\text{range } T' = (\text{null } T)^0$.

proof

For part (a), we have $\dim \text{range } T' = \dim W' - \dim \text{null } T'$
 $= \dim W - \dim (\text{range } T)^0$
 $= \cancel{\dim W} - (\cancel{\dim W} - \dim \text{range } T)$
 $= \underline{\dim \text{range } T}$

For part (b), $\varphi \in \text{range } T'$. Then, $\exists \psi \in W'$ s.t. $T'(\psi) = \varphi$. Now, if $v \in \text{null } T$, then $\varphi(v) = T'(\psi)(v) = \psi(T(v)) = \psi(0) = 0$. Hence, $\varphi \in (\text{null } T)^0$. So, $\text{range } T' \subseteq (\text{null } T)^0$.

To show $\text{range } T' = (\text{null } T)^0$, we show they have the same dimension.

Notice: $\dim \text{range } T' = \dim \text{range } T = \dim V - \dim \text{null } T = \dim (\text{null } T)^0$.

So, $\text{range } T' = (\text{null } T)^0$.

3.131 T injective is equivalent to T' surjective

Suppose V and W are finite-dimensional and $T \in \mathcal{L}(V, W)$. Then

T is injective $\iff T'$ is surjective.

proof

T is injective $\iff \text{null } T = \{0\}$
 $\iff (\text{null } T)^0 = V'$
 $\iff \text{range } T' = V'$
 $\iff T'$ is surjective.

Matrix of Dual of Linear Map

3.132 matrix of T' is transpose of matrix of T

Suppose V and W are finite-dimensional and $T \in \mathcal{L}(V, W)$. Then

$$\mathcal{M}(T') = \underline{(\mathcal{M}(T))^t}.$$

Proof If $\psi_1, \dots, \psi_m$ is a basis for W' & $\varphi_1, \dots, \varphi_n$ is a basis for W , where $\psi_1, \dots, \psi_m$ is the dual basis of $\omega_1, \dots, \omega_m$ & $\varphi_1, \dots, \varphi_n$ is the dual basis of $v_1, \dots, v_n$, then we have

$$T'(\psi_j) = \sum_{r=1}^n \mathcal{M}(T')_{r,j} \varphi_r$$

Now, apply this to v_k .

$$\psi_j(T v_k) = T'(\psi_j)(v_k) = \sum_{r=1}^n \mathcal{M}(T')_{r,j} \varphi_r(v_k) = \mathcal{M}(T')_{k,j}$$

$$\begin{aligned} \text{We also know } \psi_j(T v_k) &= \psi_j\left(\sum_{r=1}^m \mathcal{M}(T)_{r,k} \omega_r\right) \\ &= \sum_{r=1}^m \mathcal{M}(T)_{r,k} \psi_j(\omega_r) \\ &= \mathcal{M}(T)_{j,k} \end{aligned}$$

So, $\mathcal{M}(T')_{k,j} = \mathcal{M}(T)_{j,k}$ and

$$\mathcal{M}(T') = (\mathcal{M}(T))^T.$$

3.133 column rank equals row rank

Suppose $A \in \mathbb{F}^{m,n}$. Then the column rank of A equals the row rank of A .