3F Duality

Dual Space and Dual Map

3.108 definition: *linear functional*

A *linear functional* on V is a linear map from V to $\mathbf{F}$. In other words, a linear functional is an element of $\mathcal{L}(V, \mathbf{F})$.

3.110 definition: *dual space, V'*

The *dual space* of V, denoted by V', is the vector space of all linear functionals on V. In other words, $V' = \mathcal{L}(V, \mathbf{F})$.

3.111 $\dim V' = \dim V$

Suppose V is finite-dimensional. Then V' is also finite-dimensional and

$$\dim V' = \dim V.$$

$\boxed{\text{Proof}}$ We know that if V & W are finite-dim vector spaces, then $\dim \mathcal{L}(V,W) = (\dim V)(\dim W)$.
So, if $W = \mathbb{F}$, $\dim W = \dim \mathbb{F} = 1$ and we get

$$\dim V' = \dim \mathcal{L}(V, \mathbb{F}) = (\dim V)(\dim \mathbb{F}) = \dim V.$$

3.112 definition: *dual basis*

If $v_1, \ldots, v_n$ is a basis of V, then the *dual basis* of $v_1, \ldots, v_n$ is the list $\varphi_1, \ldots, \varphi_n$ of elements of V', where each φ_j is the linear functional on V such that

$$\varphi_j(v_k) = \begin{cases} 1 & \text{if } k = j, \\ 0 & \text{if } k \neq j. \end{cases}$$

Since φ_j is defined on a basis, it is defined everywhere by linearity — i.e. $\varphi_j(v) = \varphi_j(a_1 v_1 + \cdots + a_n v_n) = a_1 \varphi_j(v_1) + \cdots + a_n \varphi_j(v_n) = a_j \varphi_j(v_j) = a_j$.

3.114 *dual basis gives coefficients for linear combination*

Suppose $v_1, \ldots, v_n$ is a basis of V and $\varphi_1, \ldots, \varphi_n$ is the dual basis. Then

$$v = \varphi_1(v)v_1 + \cdots + \varphi_n(v)v_n$$

for each $v \in V$.

$\boxed{\text{proof}}$ Since $v_1, \ldots, v_n$ is a basis of V, $\exists a_1, \ldots, a_n \in \mathbb{F}$ s.t. $v = a_1 v_1 + \cdots + a_n v_n$. So, by linearity, we get

$$\varphi_j(v) = \varphi_j(a_1 v_1 + \cdots + a_n v_n) = a_1 \varphi_j(v_1) + \cdots + a_n \varphi_j(v_n) = a_j \varphi_j(v_j) = a_j$$

for each $j \in \{1, \ldots, n\}$. So, this says

$$v = a_1 v_1 + \cdots + a_n v_n = \varphi_1(v) v_1 + \cdots + \varphi_n(v) v_n.$$

3.116 *dual basis is a basis of the dual space*

Suppose V is finite-dimensional. Then the dual basis of a basis of V is a basis of V'.

$\boxed{\text{proof}}$ $\mathcal{S}$ $v_1, \ldots, v_n$ is a basis of V. Let $\varphi_1, \ldots, \varphi_n$ denote the corresponding dual basis. For lin. indep., $\mathcal{S}$ $a_1, \ldots, a_n \in \mathbb{F}$ are s.t. $a_1\varphi_1 + \cdots + a_n \varphi_n = 0$ (where $0 \in \mathcal{L}(V, \mathbb{F}) = V'$ is the map $0(v) = 0$ $\forall v \in V$).

But then, $0 = 0(v_j) = (a_1\varphi_1 + \cdots + a_n\varphi_n)(v_j) = a_1\varphi_1(v_j) + \cdots + a_n\varphi_n(v_j) = a_j\varphi_j(v_j) = a_j$.

So, since this holds for all $j \in \{1, \ldots, n\}$, this says $a_1 = \cdots = a_n = 0$.

Thus, $\varphi_1, \ldots, \varphi_n$ is a lin. indep. list in V' of length $\dim V'$, so it is a basis.

3.118 definition: *dual map, T'*

Suppose $T \in \mathcal{L}(V, W)$. The *dual map* of T is the linear map $T' \in \mathcal{L}(W', V')$ defined for each $\varphi \in W'$ by

$$T'(\varphi) = \varphi \circ T.$$

The claimed linearity is a consequence of the definition of T': it is just a composition of two linear maps!

Visualize

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[dr, dashed, "T' = \varphi \circ T"'] & W \arrow[d, "\varphi"] \\ & \mathbb{F} \end{tikzcd}$$

Notice: T' basically "flops" the domain & codomain of T around. Technically, it flops & dualizes, but since V & V' (also, W & W') are isomorphic if finite dim., this just relabeling.

$$T'(\varphi + \psi) = (\varphi + \psi) \circ T = \varphi \circ T + \psi \circ T = T'(\varphi) + T'(\psi)$$

$$T'(\lambda \varphi) = \lambda \varphi \circ T = \lambda (\varphi \circ T) = \lambda T'(\varphi)$$

3.120 *algebraic properties of dual maps*

Suppose $T \in \mathcal{L}(V, W)$. Then

(a) $(S + T)' = S' + T'$ for all $S \in \mathcal{L}(V, W)$;

(b) $(\lambda T)' = \lambda T'$ for all $\lambda \in \mathbf{F}$;

(c) $(ST)' = T'S'$ for all $S \in \mathcal{L}(W, U)$.

Proof

(a) Let $\varphi \in W'$. Then, $(S+T)'(\varphi) = \varphi \circ (S+T) = \varphi \circ S + \varphi \circ T = S'(\varphi) + T'(\varphi)$

(b) Let $\varphi \in W'$ & $\lambda \in \mathbb{F}$. Then, $(\lambda T)'(\varphi) = \varphi \circ (\lambda T) \underset{\text{since } \varphi \text{ is linear}}{=} \lambda(\varphi \circ T) = \lambda T'(\varphi)$

(c) Let $S \in \mathcal{L}(W, U)$ & $\varphi \in U'$. Then, $(ST)'(\varphi) = \varphi \circ ST = (\varphi \circ S) \circ T = T'(\varphi \circ S) = T'(S'(\varphi)) = (T'S')(\varphi)$.

Null Space and Range of Dual of Linear Map

3.121 definition: *annihilator, U^0*

For $U \subseteq V$, the *annihilator* of U, denoted by U^0, is defined by

$$U^0 = \{\varphi \in V' : \varphi(u) = 0 \text{ for all } u \in U\}.$$

3.124 *the annihilator is a subspace*

Suppose $U \subseteq V$. Then U^0 is a subspace of V'.

$\boxed{\text{Proof}}$ We need only show $0 \in U^0$, U^0 is closed under addition, & U^0 is closed under scalar multiplication.

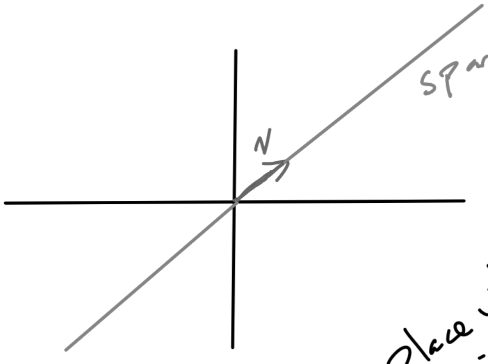
First, note that $0 \in V' = \mathcal{L}(V, \mathbb{F})$ is defined by $0(v) = 0$ $\forall v \in V$.

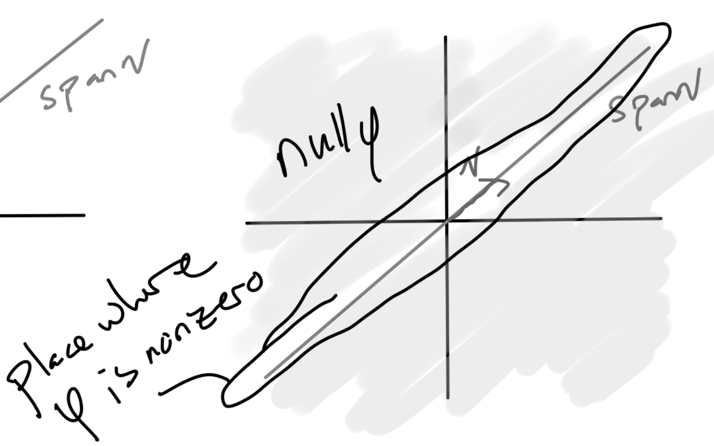
So, $0(u) = 0$ for all $u \in U$ & $0 \in U^0$.

Now, & $\varphi, \psi \in U^0$. Then, $\varphi(u) = 0$, $\psi(u) = 0$ $\forall u \in U$. Hence, we have $(\varphi + \psi)(u) = \varphi(u) + \psi(u) = 0 + 0 = 0$, so $\varphi + \psi \in U^0$ as well.

Finally, & $\varphi \in U^0$ & $\lambda \in \mathbb{F}$. Then, $(\lambda\varphi)(u) = \lambda(\varphi(u)) = \lambda(0) = 0$ $\forall u \in U$, so $\lambda\varphi \in U^0$ as well.

$\boxed{\text{Idea}}$ We can look at vectors & the subspaces they span. The idea of a dual space is to invert this:





The idea is to find something in V' that is 0 except on span v