

3E Products and Quotients of Vector Spaces

Products of Vector Spaces

3.87 definition: *product of vector spaces*

Suppose $V_1, \dots, V_m$ are vector spaces over F . $\leftarrow$ Same field!

- The *product* $V_1 \times \dots \times V_m$ is defined by

$$V_1 \times \dots \times V_m = \{(\underline{v_1}, \dots, \underline{v_m}) : \underline{v_1} \in V_1, \dots, \underline{v_m} \in V_m\}.$$

- Addition on $V_1 \times \dots \times V_m$ is defined by

$$(\underline{u_1}, \dots, \underline{u_m}) + (\underline{v_1}, \dots, \underline{v_m}) = (\underline{u_1 + v_1}, \dots, \underline{u_m + v_m}).$$

- Scalar multiplication on $V_1 \times \dots \times V_m$ is defined by

$$\underline{\lambda}(\underline{v_1}, \dots, \underline{v_m}) = (\underline{\lambda v_1}, \dots, \underline{\lambda v_m}).$$

3.89 *product of vector spaces is a vector space*

Suppose $V_1, \dots, V_m$ are vector spaces over F . Then $V_1 \times \dots \times V_m$ is a vector space over F .

3.92 *dimension of a product is the sum of dimensions*

Suppose $V_1, \dots, V_m$ are finite-dimensional vector spaces. Then $V_1 \times \dots \times V_m$ is finite-dimensional and

$$\boxed{\dim(V_1 \times \dots \times V_m)} = \underline{\dim V_1} + \dots + \underline{\dim V_m}.$$

Proof Let's construct a basis for $V_1 \times \dots \times V_m$.

Let $v_{1,k}, \dots, v_{n_k,k}$ be a basis for V_k where $\dim V_k = n_k$.
 $\dim V_1$ $\dim V_2$ $\& \text{ so on}$

We take $(\underline{v_{1,1}}, 0, \dots, 0), \dots, (\underline{v_{n_1,1}}, 0, \dots, 0), (0, \underline{v_{1,2}}, 0, \dots, 0), \dots, (0, \underline{v_{n_2,2}}, 0, \dots, 0), \dots, (0, \dots, 0, \underline{v_{1,m}}), \dots, (0, \dots, 0, \underline{v_{n_m,m}})$

(i.e. take the list of vectors in $V_1 \times \dots \times V_m$ where everything is 0 except for one component, which is a vector in a basis of the corresponding space).

Since $v_k \in V_k$ can be written uniquely as $v_k = a_1 v_{1,k} + \dots + a_{n_k} v_{n_k,k}$, then

$(0, \dots, 0, v_k, 0, \dots, 0) \in V_1 \times \dots \times V_m$ can be written uniquely as a linear combo of

we can write every $(v_1, \dots, v_m) \in V_1 \times \dots \times V_m$ uniquely as a linear combo of by breaking apart $(v_1, \dots, v_m) = (v_1, 0, \dots, 0) + \dots + (0, \dots, 0, v_m)$. Hence, $\dim V_1 \times \dots \times V_m$ is the sum of the dimensions of the V_i 's.

3.93 products and direct sums

Suppose that $V_1, \dots, V_m$ are subspaces of $\underline{V}$. Define a linear map $\Gamma : \underline{V_1 \times \dots \times V_m} \rightarrow \underline{V_1 + \dots + V_m}$ by

$$\Gamma(v_1, \dots, v_m) = v_1 + \dots + v_m.$$

Then $\underline{V_1 + \dots + V_m}$ is a direct sum if and only if Γ is injective.

Proof If Γ is injective, then $\Gamma(v_1, \dots, v_m) = 0 \Rightarrow (v_1, \dots, v_m) = 0$. So, the only way to write 0 as a sum of the form $v_1 + \dots + v_m = 0$ with $v_k \in V_k$ is $v_1 = \dots = v_m = 0$, which says $V_1 + \dots + V_m$ is direct. Now, if $V_1 \oplus \dots \oplus V_m$, then Γ must be injective, since directness implies that $v_1 + \dots + v_m = 0$ for $v_k \in V_k$ means $v_1 = \dots = v_m = 0$, so $\text{null } \Gamma = \{0\}$ & Γ is injective.

3.94 a sum is a direct sum if and only if dimensions add up

Suppose $\underline{V}$ is finite-dimensional and $V_1, \dots, V_m$ are subspaces of V . Then $\underline{V_1 + \dots + V_m}$ is a direct sum if and only if

$$\underline{\dim(V_1 + \dots + V_m) = \dim V_1 + \dots + \dim V_m}.$$

← Notice: no products!

Proof First, note that Γ (from the previous theorem) is surjective. By the FTOLM, Γ is injective if & only if $\dim(V_1 + \dots + V_m) = \dim(V_1 \times \dots \times V_m)$ (since $\dim V_1 \times \dots \times V_m = \dim \text{null } \Gamma + \dim \text{range } \Gamma = \dim \text{null } \Gamma + \dim(V_1 + \dots + V_m)$).

Now, by the previous result, Γ is injective if & only if $V_1 + \dots + V_m$ is direct. So, since $\dim(V_1 \times \dots \times V_m) = \dim V_1 + \dots + \dim V_m$, we have $\dim(V_1 + \dots + V_m) = \dim V_1 + \dots + \dim V_m$ if & only if this sum is direct.

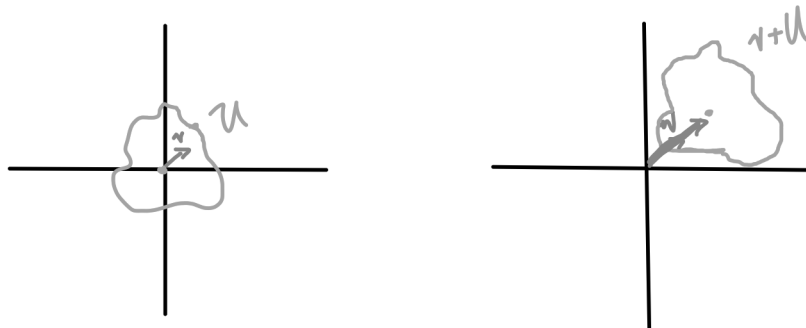
Quotient Spaces

3.95 notation: $v + U$

Suppose $\underline{v} \in V$ and $\underline{U} \subseteq V$. Then $\underline{v} + \underline{U}$ is the subset of V defined by

$$v + U = \{\underline{v} + \underline{u} : u \in U\}.$$

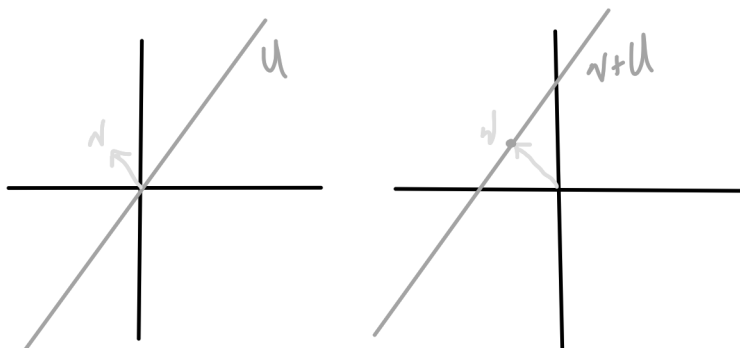
Visualize



3.97 definition: *translate*

For $v \in V$ and U a subset of V , the set $v + U$ is said to be a *translate* of U .

Visualize

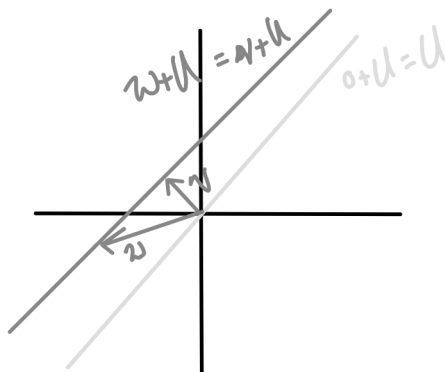


3.99 definition: *quotient space*, V/U

Suppose $\underline{U}$ is a subspace of V . Then the quotient space V/U is the set of all translates of U . Thus

$$\underline{V}/\underline{U} = \{\underline{v} + \underline{U} : \underline{v} \in \underline{V}\}.$$

Visualize



3.101 two translates of a subspace are equal or disjoint

Suppose U is a subspace of V and $v, w \in V$. Then

$$\underline{v-w \in U} \iff \underline{v+U = w+U} \iff \underline{(v+U) \cap (w+U) \neq \emptyset.}$$

Proof $\S$ $v-w \in U$. If $u \in U$, then observe that

$$v+u \Rightarrow v+u = v+u+w-w = w + ((v-w)+u) \in w+U. \text{ So, } v+U \subseteq w+U.$$

likewise, repeating the argument w/ $v \nleftrightarrow w$ interchanged, $w+U \subseteq v+U$.

$$\text{So, } v-w \in U \Rightarrow v+U = w+U.$$

$$\text{Now, } \S \ v+U = w+U. \text{ Then, } (v+U) \cap (w+U) = v+U = w+U \neq \emptyset.$$

$$\text{So, } v+U = w+U \Rightarrow (v+U) \cap (w+U) \neq \emptyset.$$

Finally, $\S \ (v+U) \cap (w+U) \neq \emptyset$. Then, $\exists u_1, u_2 \in U$ st.

$$v+u_1 = w+u_2. \text{ Rearranging, this says } v-w = u_2 - u_1 \in U.$$

$$\text{So, } (v+U) \cap (w+U) \neq \emptyset \Rightarrow v-w \in U.$$

3.102 definition: addition and scalar multiplication on V/U

Suppose U is a subspace of V . Then addition and scalar multiplication are defined on V/U by

$$(\underline{v} + U) + (\underline{w} + U) = \underline{(v + w)} + U$$

$$\lambda(\underline{v} + U) = \underline{(\lambda v)} + U$$

for all $v, w \in V$ and all $\lambda \in \mathbb{F}$.

3.103 quotient space is a vector space

Suppose U is a subspace of V . Then V/U , with the operations of addition and scalar multiplication as defined above, is a vector space.

Proof The problem we want to show does not happen is the following: If $v_1 + U = v_2 + U$ & $w_1 + U = w_2 + U$, then could we get $(v_1 + w_1) + U \neq (v_2 + w_2) + U$?

We show this never happens. $\S$ $v_1 + U = v_2 + U$ & $w_1 + U = w_2 + U$. Then, $v_1 - v_2 \in U$ & $w_1 - w_2 \in U$. Now, this says, since U is a subspace, that $(v_1 - v_2) + (w_1 - w_2) \in U$. So, this says $(v_1 + w_1) - (v_2 + w_2) \in U$. But, we just showed that this implies $(v_1 + w_1) + U = (v_2 + w_2) + U$. So, addition makes sense!

Likewise, $\S$ $\lambda \in \mathbb{F}$ & $v_1 + U = v_2 + U$. Since U is a subspace, it's closed under scaling, so $v_1 - v_2 \in U$ implies $\lambda(v_1 - v_2) \in U$.

Hence, $\lambda v_1 - \lambda v_2 \in U$ & $\lambda v_1 + U = \lambda v_2 + U$. So, scalar mult. makes sense too!

With this, the rest is routine (exercise).

3.104 definition: *quotient map*, π

Suppose $\underline{U}$ is a subspace of $\underline{V}$. The *quotient map* $\underline{\pi}: \underline{V} \rightarrow \underline{V/U}$ is the linear map defined by

$$\pi(v) = v + U$$

for each $v \in V$.

3.105 *dimension of quotient space*

Suppose V is finite-dimensional and $\underline{U}$ is a subspace of $\underline{V}$. Then

$$\underline{\dim V/U} = \underline{\dim V} - \underline{\dim U}. \quad \leftarrow$$

Proof Let π be the quotient map. Then,
since $\pi(v) = v + U$ & $V/U = \{v + U : v \in V\}$, π is surjective.

Now, notice that $\pi(v) = 0 + U$ (i.e. $v \in \text{null } \pi$), then
 $v + U = 0 + U$, which means $v = v - 0 \in U$. So, $\text{null } \pi = U$.

By the FTOLM, $\dim V = \dim \text{range } \pi + \dim \text{null } \pi$
 $\dim V = \dim V/U + \dim U$
 $\Rightarrow \dim V/U = \dim V - \dim U.$

3.106 notation: $\tilde{T}$

Suppose $T \in \mathcal{L}(V, W)$. Define $\tilde{T}: V/(\text{null } T) \rightarrow W$ by

$$\tilde{T}(v + \text{null } T) = Tv.$$

This makes sense: If $v + \text{null } T = u + \text{null } T$, then
 $v - u \in \text{null } T \nmid T(v - u) = 0$, i.e. $Tv = Tu$.

So, if $v + \text{null } T = u + \text{null } T$, $\tilde{T}(v + \text{null } T) = Tv = Tu = \tilde{T}(u + \text{null } T)$

That is, $\tilde{T}$ does not depend on the choice of v used to represent $v + \text{null } T$.

3.107 null space and range of $\tilde{T}$

Suppose $T \in \mathcal{L}(V, W)$. Then

(a) $\tilde{T} \circ \pi = T$, where π is the quotient map of V onto $V/(\text{null } T)$;

(b) $\tilde{T}$ is injective;

(c) $\text{range } \tilde{T} = \text{range } T$;

(d) $V/(\text{null } T)$ and $\text{range } T$ are isomorphic vector spaces.

$\pi: V \rightarrow V/\text{null } T$

Proof

(a) If $v \in V$, then $(\tilde{T} \circ \pi)(v) = \tilde{T}(\pi(v)) = \tilde{T}(v + \text{null } T) = Tv$

So, $\tilde{T} \circ \pi = T$

(b) We show injectivity by showing $\text{null } \tilde{T}$ is trivial
 (i.e. $\text{null } \tilde{T} = \{0 + \text{null } T\}$). $\nmid \tilde{T}(v + \text{null } T) = 0$. Then,
 $Tv = 0$, since $\tilde{T}(v + \text{null } T) = Tv$. So, $v \in \text{null } T \nmid$
 $v + \text{null } T = 0 + \text{null } T$ since $v - 0 \in \text{null } T$. Hence, $\tilde{T}$ is inj.

(c) If $w \in \text{range } T$, then $\exists v \in V$ s.t. $Tv = w$. So,

$\tilde{T}(v + \text{null } T) = Tv = w \nmid w \in \text{range } \tilde{T}$ as well.

Write this argument in reverse to get $w \in \text{range } T$ whenever $w \in \text{range } \tilde{T}$.

(d) From (b) & (c), the map $R: V/\text{null } T \rightarrow \text{range } T$ defined by
 $R(v + \text{null } T) = Tv$ is the same as $\tilde{T}$ but with a different codomain. So, R is inj. & surject.