Recall let $A = \mathcal{M}(T)$. Then, $Tv_k = \sum_{j=1}^{m} A_{jk} w_j$

3.71 *$\mathcal{L}(V,W)$ and $\mathbf{F}^{m,n}$ are isomorphic*

Suppose $v_1, \ldots, v_n$ is a basis of V and $w_1, \ldots, w_m$ is a basis of W. Then $\mathcal{M}$ is an isomorphism between $\mathcal{L}(V,W)$ and $\mathbf{F}^{m,n}$.

i.e. $\dim W = m$

i.e. $\mathbb{F}^{\dim W, \dim V}$

i.e. $\dim V = n$

$\boxed{\text{Proof}}$ From a previous section, we know $\mathcal{M}$ is linear. So, to show $\mathcal{M}$ is an isomorphism, it is enough to show that $\mathcal{M}$ is injective & surjective. For injectivity, if $T \in \mathcal{L}(V,W)$ is s.t. $\mathcal{M}(T) = 0$. Then, $Tv_k = 0$ for $k = 1, \ldots, n$. Since $v_1, \ldots, v_n$ is a basis for V, this says $T = 0$. Hence, $\operatorname{null} \mathcal{M} = \{0\}$ & $\mathcal{M}$ is injective. For surjectivity, the linear map lemma says for any $A \in \mathbb{F}^{m,n}$, there exists $T \in \mathcal{L}(V,W)$ s.t. $Tv_k = \sum_{j=1}^{m} A_{jk} w_j$ for $k = 1, \ldots, n$. But this is just $\mathcal{M}(T)$, so $\mathcal{M}$ is surjective. Hence, it is an isomorphism.

3.72 $\dim \mathcal{L}(V,W) = (\dim V)(\dim W)$

Suppose V and W are finite-dimensional. Then $\mathcal{L}(V,W)$ is finite-dimensional and

$$\dim \mathcal{L}(V,W) = (\dim V)(\dim W).$$

$\boxed{\text{Proof}}$ By the previous result, $\mathcal{L}(V,W)$ & $\mathbb{F}^{\dim W, \dim V}$ are isomorphic. But, by the last result from last time, this means they have the same dimension. So, $\dim \mathcal{L}(V,W) = \dim \mathbb{F}^{\dim W, \dim V} = \dim W \dim V$.

Linear Maps Thought of as Matrix Multiplication

3.73 definition: *matrix of a vector,* $\mathcal{M}(v)$

Suppose $v \in V$ and $v_1, \ldots, v_n$ is a basis of V. The *matrix of v* with respect to this basis is the n-by-1 matrix i.e. $\dim V = n$

$$\mathcal{M}(v) = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix},$$

where $b_1, \ldots, b_n$ are the scalars such that

$$v = b_1 v_1 + \cdots + b_n v_n.$$

3.75 $\mathcal{M}(T)_{\cdot,k} = \mathcal{M}(Tv_k)$.

Suppose $T \in \mathcal{L}(V, W)$ and $v_1, \ldots, v_n$ is a basis of V and $w_1, \ldots, w_m$ is a basis of W. Let $1 \le k \le n$. Then the k^{th} column of $\mathcal{M}(T)$, which is denoted by $\mathcal{M}(T)_{\cdot,k}$, equals $\mathcal{M}(Tv_k)$.

3.76 *linear maps act like matrix multiplication*

Suppose $T \in \mathcal{L}(V, W)$ and $v \in V$. Suppose $v_1, \ldots, v_n$ is a basis of V and $w_1, \ldots, w_m$ is a basis of W. Then

$$\mathcal{M}(Tv) = \mathcal{M}(T)\mathcal{M}(v).$$

$\boxed{\text{Proof}}$ Suppose $v = b_1 v_1 + \cdots + b_n v_n$ for $b_1, \ldots, b_n \in \mathbb{F}$. Then,

$Tv = b_1 Tv_1 + \cdots + b_n Tv_n$. Hence,

$$\begin{aligned} \mathcal{M}(Tv) &= b_1 \mathcal{M}(Tv_1) + \cdots + b_n \mathcal{M}(Tv_n) \\ &= b_1 \mathcal{M}(T)_{\cdot,1} + \cdots + b_n \mathcal{M}(T)_{\cdot,n} \\ &= \begin{bmatrix} \mathcal{M}(T)_{\cdot,1} & \cdots & \mathcal{M}(T)_{\cdot,n} \end{bmatrix} \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} \\ &= \mathcal{M}(T)\mathcal{M}(v). \end{aligned}$$

3.78 *dimension of* range T *equals column rank of* $\mathcal{M}(T)$

Suppose V and W are finite-dimensional and $T \in \mathcal{L}(V, W)$. Then dim range T equals the column rank of $\mathcal{M}(T)$.

$\boxed{\text{Proof}}$ S $v_1, \ldots, v_n$ is a basis for V & $w_1, \ldots, w_m$ is a basis for W. The map $\mathcal{M} \in \mathcal{L}(W, \mathbb{F}^{m,1})$ that takes $w \in W$ to $\mathcal{M}(w) = \begin{bmatrix} a_1 \\ \vdots \\ a_m \end{bmatrix}$ where $w = a_1 w_1 + \cdots + a_m w_m$ is an isomorphism. Now, since range $T = \operatorname{span}(Tv_1, \ldots, Tv_n)$ for $T \in \mathcal{L}(V,W)$, take $\mathcal{M}\big|_{\text{range } T}$. This is an isomorphism from range T to $\operatorname{span}(\mathcal{M}(Tv_1), \ldots, \mathcal{M}(Tv_n))$. But, $\mathcal{M}(Tv_k) = \mathcal{M}(T)_{\cdot,k}$, so this says that range T is isomorphic to $\operatorname{span}(\mathcal{M}(T)_{\cdot,1}, \ldots, \mathcal{M}(T)_{\cdot,n})$, i.e. the range of T is the span of the columns of $\mathcal{M}(T)$. So, they have the same dimension, which is the rank of $\mathcal{M}(T)$.

Change of Basis

3.81 *matrix of product of linear maps*

Suppose $T \in \mathcal{L}(U, V)$ and $S \in \mathcal{L}(V, W)$. If $u_1, \ldots, u_m$ is a basis of U, $v_1, \ldots, v_n$ is a basis of V, and $w_1, \ldots, w_p$ is a basis of W, then

$$\mathcal{M}(ST, (u_1, \ldots, u_m), (w_1, \ldots, w_p)) = \underbrace{\mathcal{M}(S, (v_1, \ldots, v_n), (w_1, \ldots, w_p))\mathcal{M}(T, (u_1, \ldots, u_m), (v_1, \ldots, v_n))}.$$

$\underbrace{\mathcal{M}(ST) = \mathcal{M}(S)\mathcal{M}(T)}$

doesn't explicitly show what base is being used in the middle

3.82 *matrix of identity operator with respect to two bases*

Suppose that $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$ are bases of V. Then the matrices

$$\mathcal{M}(I, (u_1, \ldots, u_n), (v_1, \ldots, v_n)) \quad \text{and} \quad \mathcal{M}(I, (v_1, \ldots, v_n), (u_1, \ldots, u_n))$$

are invertible, and each is the inverse of the other.

3.84 *change-of-basis formula*

Suppose $T \in \mathcal{L}(V)$. Suppose $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$ are bases of V. Let

$$A = \mathcal{M}(T, (u_1, \ldots, u_n)) \quad \text{and} \quad B = \mathcal{M}(T, (v_1, \ldots, v_n))$$

and $C = \mathcal{M}(I, (u_1, \ldots, u_n), (v_1, \ldots, v_n))$. Then

$$A = C^{-1}BC.$$

$\boxed{\text{Proof}}$ Recall the following:

$$\boxed{\begin{array}{l} \mathcal{M}(ST, (u_1, \ldots, u_m), (w_1, \ldots, w_p)) = \mathcal{M}(S, (v_1, \ldots, v_n), (w_1, \ldots, w_p))\, \mathcal{M}(T, (u_1, \ldots, u_m), (v_1, \ldots, v_n)) \\ \text{for } S \in \mathcal{L}(V, W),\ T \in \mathcal{L}(U, V),\ u_1, \ldots, u_m \text{ a basis for } U,\ v_1, \ldots, v_n \text{ a basis for } V, \\ \text{and } w_1, \ldots, w_p \text{ a basis for } W \end{array}}$$

Since $C^{-1} = \mathcal{M}(I, (v_1, \ldots, v_n), (u_1, \ldots, u_n))$, we have

$$\left.\begin{aligned} C^{-1}BC &= \mathcal{M}(I, v, u)\, \mathcal{M}(T, v, v)\, \mathcal{M}(I, u, v) \\ &= \mathcal{M}(IT, v, u)\, \mathcal{M}(I, u, v) \\ &= \mathcal{M}(ITI, u, u) \\ &= \mathcal{M}(T, u, u) \\ &= A \end{aligned}\right\}$$

here, $u = (u_1, \ldots, u_n)$
$v = (v_1, \ldots, v_n)$
for shorthand

3.86 *matrix of inverse equals inverse of matrix*

Suppose that $v_1, \ldots, v_n$ is a basis of V and $T \in \mathcal{L}(V)$ is invertible. Then $\mathcal{M}(T^{-1}) = (\mathcal{M}(T))^{-1}$, where both matrices are with respect to the basis $v_1, \ldots, v_n$.

$\boxed{\text{Proof}}$ Hint: $\mathcal{M}(T)^{-1}$ satisfies $\mathcal{M}(T)\mathcal{M}(T)^{-1} = I$. Could we use some bases for V in terms of the matrix of T & the matrix of its inverse?