

3D Invertibility and Isomorphisms

Invertible Linear Maps

3.59 definition: *invertible, inverse*

- A linear map $T \in \mathcal{L}(\underline{V}, \underline{W})$ is called invertible if there exists a linear map $S \in \mathcal{L}(\underline{W}, \underline{V})$ such that $\underline{ST}$ equals the identity operator on $\underline{V}$ and $\underline{TS}$ equals the identity operator on $\underline{W}$.
- A linear map $S \in \mathcal{L}(\underline{W}, \underline{V})$ satisfying $\underline{ST} = \underline{I}_V$ and $\underline{TS} = \underline{I}_W$ is called an inverse of $\underline{T}$ (note that the first I is the identity operator on V and the second I is the identity operator on W).

3.60 *inverse is unique*

An invertible linear map has a unique inverse.

Proof $\S$ $T \in \mathcal{L}(\underline{V}, \underline{W})$ is invertible. Then, $\exists S \in \mathcal{L}(\underline{W}, \underline{V})$ s.t. $\underline{TS} = \underline{I}_W$ & $\underline{ST} = \underline{I}_V$. Suppose $S' \in \mathcal{L}(\underline{W}, \underline{V})$ satisfies $\underline{TS'} = \underline{I}_W$ & $\underline{S'T} = \underline{I}_V$. Then, we have

$$S = S\underline{I}_W = S(\underline{TS'}) = (\underline{ST})S' = \underline{I}_V S' = S'$$

So, $S = S'$ & inverses are unique!

3.61 notation: T^{-1}

If $\underline{T}$ is invertible, then its inverse is denoted by $\underline{T}^{-1}$. In other words, if $\underline{T} \in \mathcal{L}(\underline{V}, \underline{W})$ is invertible, then $\underline{T}^{-1}$ is the unique element of $\mathcal{L}(\underline{W}, \underline{V})$ such that $\underline{T}^{-1}\underline{T} = \underline{I}_V$ and $\underline{T}\underline{T}^{-1} = \underline{I}_W$.

3.63 invertibility $\iff$ injectivity and surjectivity

A linear map is invertible if and only if it is injective and surjective.

Proof First, assume $T \in \mathcal{L}(V, W)$ is invertible. Then, observe that if $w \in W$, then $T(T^{-1}w) = Iw = w$. So, T is surjective. For injectivity, $\exists u, v \in V$ are s.t. $Tu = Tv$.

Then, $u = T^{-1}(Tu) = T^{-1}(Tv) = v$, so T is injective.

Now, $\exists T \in \mathcal{L}(V, W)$ is injective & surjective. For $w \in W$, define S_w to be the unique element of V s.t. $TS_w = w$.

This can be done because: T is surjective, so $\exists v \in V$ s.t. $Tv = w$ for whatever w we chose, and, since T is injection, if $Tv = w = Tu$, then $v = u$ (so, this choice is unique & S is well defined). By construction, $TS = I_W$.

Now, observe that $T(STv) = (TS)(Tv) = Tw$. Since T is injective, $T(STv) = Tw$ implies $STv = v$. So, $ST = I_V$.

All that remains is to show $S \in \mathcal{L}(W, V)$. $\exists w_1, w_2 \in W$.

Then, $T(Sw_1 + Sw_2) = TS_{w_1} + TS_{w_2} = w_1 + w_2$. Thus,

$Sw_1 + Sw_2$ is the unique element that T maps to $w_1 + w_2$.

By the definition of S , this says $S(w_1 + w_2) = Sw_1 + Sw_2$.

Similarly, let $\lambda \in \mathbb{F}$. Then, $T(\lambda Sw) = \lambda TS_w = \lambda w$, and

$S(\lambda w) = \lambda Sw$ by the definition of S . So, $S \in \mathcal{L}(W, V)$

That is, T is invertible & $S = T^{-1}$.

3.65 injectivity is equivalent to surjectivity (if $\dim V = \dim W < \infty$)

Suppose that V and W are finite-dimensional vector spaces, $\dim V = \dim W$, and $T \in \mathcal{L}(V, W)$. Then

$$T \text{ is invertible} \iff T \text{ is injective} \iff T \text{ is surjective.}$$

Proof

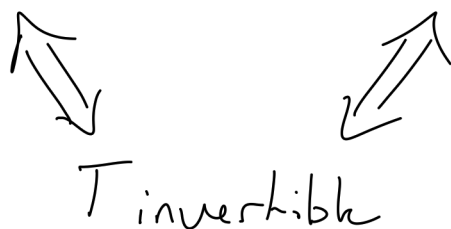
By the FTOLM, we have

$$\dim V = \dim \text{null } T + \dim \text{range } T$$

Now, if T is injective, $\text{null } T = \{0\}$ & this says $\dim V = \dim \text{range } T$ and, since $\dim V = \dim W$, $\dim \text{range } T = \dim W$. Hence, $\text{range } T = W$, so T is surjective as well. By the previous result, it is also invertible.

Likewise, if T is surjective, $\text{range } T = W$ & we have $0 = \dim V - \dim \text{range } T = \dim \text{null } T$. So, $\text{null } T = \{0\}$ & T is injective. By the previous result, it is also invertible.

$$T \text{ injective} \iff T \text{ surjective}$$



$\bar{E}_x$ Define $T \in \mathcal{L}(\mathcal{P}(\mathbb{R}))$ by $(T_p)(x) = ((x^2 + 5x + 7)p(x))''$.

We will show that for any $q \in \mathcal{P}(\mathbb{R})$, there is some solution $p \in \mathcal{P}(\mathbb{R})$ to $T_p = q$. Let $m = \deg q$. Let

$T' \in \mathcal{L}(\mathcal{P}_m(\mathbb{R}))$ be given by $(T'_p)(x) = ((x^2 + 5x + 7)p(x))''$.

Notice that if a polynomial has a second derivative of zero, then it is a poly of degree at most 1. Since multiplying a degree k poly by $x^2 + 5x + 7$ produces a degree $k+2$ poly, $T'_p = 0$ only when $p = 0$. So, T' is injective. Hence, invertible. So, $p = (T')^{-1}q$ & since T & T' agree, $T(T')^{-1}q = q$ as well.

3.68 $ST = I \iff TS = I$ (on vector spaces of the same dimension)

Suppose V and W are finite-dimensional vector spaces of the same dimension, $S \in \mathcal{L}(V, W)$, and $T \in \mathcal{L}(W, V)$. Then $ST = I$ if and only if $TS = I$.

$\dim V = \dim W$

Proof First, $\S ST = I$. If $v \in V$ & $Tv = 0$,

then $v = Iv = (ST)v = S(Tv) = S(0) = 0$. So, T is injective (since $\text{null } T = \{0\}$). Since $\dim V = \dim W$, T is invertible. So, $T^{-1} = IT^{-1} = (ST)T^{-1} = S(TT^{-1}) = SI = S$ & $S = T^{-1}$. Thus, $TS = TT^{-1} = I$. Now, for the other direction, just reverse the roles of S & T in the previous argument.

Isomorphic Vector Spaces

3.69 definition: *isomorphism, isomorphic*

- An *isomorphism* is an invertible linear map.
- Two vector spaces are called isomorphic if there is an isomorphism from one vector space onto the other one.

3.70 *dimension shows whether vector spaces are isomorphic*

Two finite-dimensional vector spaces over F are isomorphic if and only if they have the same dimension.

proof $\S$ V & W are isomorphic. Then, $\exists T \in \mathcal{L}(V, W)$ that is invertible and, hence, both injective & surjective. So, $\text{null } T = \{0\}$ & $\text{range } T = W$. The FTOLM then becomes

$$\dim V = \dim \text{null } T + \dim \text{range } T = 0 + \dim W = \dim W.$$

$\S$ $\dim V = \dim W < \infty$. Let $v_1, \dots, v_n$ be a basis for V & let $w_1, \dots, w_n$ be a basis for W (i.e. $n = \dim V = \dim W$).

Define $T \in \mathcal{L}(V, W)$ by $Tv = T(c_1v_1 + \dots + c_nv_n) = c_1w_1 + \dots + c_nw_n$ (where $c_1, \dots, c_n \in F$ are the unique coefficients in the linear combo of $v_1, \dots, v_n$ s.t. $v = c_1v_1 + \dots + c_nv_n$). T is surjective because $w_1, \dots, w_n$ spans W . Furthermore, $Tv = 0 \Leftrightarrow c_1w_1 + \dots + c_nw_n = 0$ which, by linear independence, implies $c_1 = \dots = c_n = 0$ & $v = 0v_1 + \dots + 0v_n = 0$. So, T is injective. Since T is both injective & surjective, it is an isomorphism (i.e. invertible).