

3.44 notation: $A_{j,\cdot}$, $A_{\cdot,k}$

Suppose A is an m -by- n matrix.

- If $1 \leq j \leq m$, then $A_{j,\cdot}$ denotes the 1-by- n matrix consisting of row j of A .
- If $1 \leq k \leq n$, then $A_{\cdot,k}$ denotes the m -by-1 matrix consisting of column k of A .

3.46 entry of matrix product equals row times column

Suppose A is an m -by- n matrix and B is an n -by- p matrix. Then

$$(AB)_{j,k} = A_{j,\cdot} B_{\cdot,k}$$

if $1 \leq j \leq m$ and $1 \leq k \leq p$. In other words, the entry in row j , column k , of AB equals (row j of A) times (column k of B).

Proof By definition, $(AB)_{j,k} = A_{j,1} B_{1,k} + \dots + A_{j,n} B_{n,k}$

$$= \begin{bmatrix} A_{j,1} & \dots & A_{j,n} \end{bmatrix} \begin{bmatrix} B_{1,k} \\ \vdots \\ B_{n,k} \end{bmatrix}$$
$$= A_{j,\cdot} B_{\cdot,k}$$

3.48 column of matrix product equals matrix times column

Suppose A is an m -by- n matrix and B is an n -by- p matrix. Then

$$(AB)_{\cdot,k} = A B_{\cdot,k}$$

if $1 \leq k \leq p$. In other words, column k of AB equals A times column k of B .

Proof By the previous result, $(AB)_{j,k} = A_{j,\cdot} B_{\cdot,k}$.

$$\text{So, } (AB)_{\cdot,k} = A_{\cdot,\cdot} B_{\cdot,k} = A B_{\cdot,k}.$$

3.50 linear combination of columns

Suppose A is an m -by- n matrix and $b = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$ is an n -by-1 matrix. Then

$$Ab = b_1 A_{.,1} + \dots + b_n A_{.,n} \quad \leftarrow$$

In other words, Ab is a linear combination of the columns of A , with the scalars that multiply the columns coming from b .

Proof By the previous result, $(AB)_{.,k} = A B_{.,k}$.

But, if $B = b$, we have $Ab = (Ab)_{.,1} = A b_{.,1}$.

This says $[A_{.,1} \dots A_{.,n}] \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} = b_1 A_{.,1} + \dots + b_n A_{.,n}$.

3.51 matrix multiplication as linear combinations of columns

Suppose C is an m -by- c matrix and R is a c -by- n matrix.

(a) If $k \in \{1, \dots, n\}$, then column k of CR is a linear combination of the columns of C , with the coefficients of this linear combination coming from column k of R .

(b) If $j \in \{1, \dots, m\}$, then row j of CR is a linear combination of the rows of R , with the coefficients of this linear combination coming from row j of C .

← Same logic ↓

Proof $\forall k \in \{1, \dots, n\}$. Then, column k of CR is $(CR)_{.,k}$, is equal to a linear combo of the columns of C w/ coefficients coming from $R_{.,k}$ (i.e. the k^{th} column of R) by the previous result: $(CR)_{.,k} = C(R_{.,k}) = [C_{.,1} \dots C_{.,c}] \begin{bmatrix} R_{1,k} \\ \vdots \\ R_{c,k} \end{bmatrix} = R_{1,k} C_{.,1} + \dots + R_{c,k} C_{.,c}$

Column-Row Factorization and Rank of a Matrix

3.52 definition: *column rank, row rank*

Suppose A is an m -by- n matrix with entries in $\mathbb{F}$.

- The *column rank* of A is the dimension of the span of the columns of A in $\mathbb{F}^{m,1}$.
- The *row rank* of A is the dimension of the span of the rows of A in $\mathbb{F}^{1,n}$.

i.e. column rank A
 $\dim \text{Span}(A_{\cdot,1}, \dots, A_{\cdot,n})$

EX

Suppose

$$A = \begin{pmatrix} 4 & 7 & 1 & 8 \\ 3 & 5 & 2 & 9 \end{pmatrix}.$$

The column rank of A is the dimension of

$$\dim \left(\text{span} \left(\begin{pmatrix} 4 \\ 3 \end{pmatrix}, \begin{pmatrix} 7 \\ 5 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 8 \\ 9 \end{pmatrix} \right) \right) = 2$$

The row rank of A is the dimension of

$$\dim \left(\text{span} \left((4 \ 7 \ 1 \ 8), (3 \ 5 \ 2 \ 9) \right) \right) = 2$$

These seem to be the same!

Is this always so?

3.54 definition: *transpose, A^t*

The *transpose* of a matrix A , denoted by A^t , is the matrix obtained from A by interchanging rows and columns. Specifically, if A is an m -by- n matrix, then A^t is the n -by- m matrix whose entries are given by the equation

$$(A^t)_{k,j} = A_{j,k}.$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^t = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

3.56 column-row factorization

Suppose A is an m -by- n matrix with entries in $\mathbb{F}$ and column rank $c \geq 1$. Then there exist an m -by- c matrix C and a c -by- n matrix R , both with entries in $\mathbb{F}$, such that $A = CR$.

Proof The list $A_{\cdot,1}, \dots, A_{\cdot,n}$ of columns of A can be reduced to a basis of $\text{span}(A_{\cdot,1}, \dots, A_{\cdot,n})$ (i.e. a spanning list in a finite-dim. V.S. can be reduced to a basis). This basis has length $\dim(\text{span}(A_{\cdot,1}, \dots, A_{\cdot,n})) = c$. The c columns that form this basis can be used to make a matrix C which has these basis elements as its columns. This matrix is m -by- c . Since any column in A can be written as a linear combo of the columns of C , let R be the matrix whose k th column is the coefficients such that $A_{\cdot,k} = R_{1,k}C_{\cdot,1} + \dots + R_{c,k}C_{\cdot,c}$. Then, $A = CR$.

3.57 column rank equals row rank

Suppose $A \in \mathbb{F}^{m,n}$. Then the column rank of A equals the row rank of A .

Proof Let c denote the column rank of A . Let $A = CR$ be the factorization assured by the previous theorem. Every row of A is a linear combo of R 's rows, and the coefficients in this linear combo come from C 's rows. Since R has c rows, this says that the row rank of A is less than or equal to c , its column rank. Now, do the same to A^t and note that:
 $\text{Column rank } A = \text{row rank } A^t \leq \text{column rank } A^t = \text{row rank } A$.
 So, $\text{column rank } A \geq \text{row rank } A$ & $\text{column rank } A \leq \text{row rank } A$.
 Hence, they are equal!

3.58 definition: rank

The rank of a matrix $A \in \mathbb{F}^{m,n}$ is the column rank of A . (or, equivalently, the row rank).