3C Matrices

Representing a Linear Map by a Matrix

3.29 definition: *matrix, $A_{j,k}$*

Suppose m and n are nonnegative integers. An m-by-n *matrix* A is a rectangular array of elements of $\mathbf{F}$ with m rows and n columns:

$$A = \begin{pmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & & \vdots \\ A_{m,1} & \cdots & A_{m,n} \end{pmatrix}.$$

The notation $A_{j,k}$ denotes the entry in row j, column k of A.

3.31 definition: *matrix of a linear map, $\mathcal{M}(T)$*

Suppose $T \in \mathcal{L}(V, W)$ and $v_1, \ldots, v_n$ is a basis of V and $w_1, \ldots, w_m$ is a basis of W. The *matrix of* T with respect to these bases is the m-by-n matrix $\mathcal{M}(T)$ whose entries $A_{j,k}$ are defined by

in W

$$Tv_k = \underbrace{A_{1,k}w_1 + \cdots + A_{m,k}w_m}_{\text{lin. combo. of a basis of } W}.$$

If the bases $v_1, \ldots, v_n$ and $w_1, \ldots, w_m$ are not clear from the context, then the notation $\mathcal{M}(T, (v_1, \ldots, v_n), (w_1, \ldots, w_m))$ is used.

$\boxed{\text{Idea}}$

$$\mathcal{M}(T) = \begin{array}{c} \\ w_1 \\ \vdots \\ w_m \end{array} \begin{array}{c} \begin{array}{ccccc} v_1 & \cdots & v_k & \cdots & v_n \end{array} \\ \begin{pmatrix} & & A_{1,k} & & \\ & & \vdots & & \\ & & A_{m,k} & & \end{pmatrix} \end{array}.$$

The k^{th} column of $\mathcal{M}(T)$ consists of the scalars needed to write Tv_k as a linear combination of $w_1, \ldots, w_m$:

$$Tv_k = \sum_{j=1}^{m} A_{j,k} w_j.$$

$\boxed{Ex}$

Suppose $T \in \mathcal{L}(\mathbf{F}^2, \mathbf{F}^3)$ is defined by

$$T(x, y) = (x + 3y, 2x + 5y, 7x + 9y).$$

With respect to the basis $(1,0),(0,1)$ of $\mathbb{F}^2$ and the basis $(1,0,0),(0,1,0),(0,0,1)$ of $\mathbb{F}^3$,

$M(T) = \begin{bmatrix} 1 & 3 \\ 2 & 5 \\ 7 & 9 \end{bmatrix}$ Since $T(1,0) = (1+3(0), 2(1)+5(0), 7(1)+9(0))$

$= (1, 2, 7) = 1(1,0,0) + 2(0,1,0) + 7(0,0,1)$

$T(0,1) = (3, 5, 9)$

$= 3(1,0,0) + 5(0,1,0) + 9(0,0,1)$

$\boxed{Ex}$

Suppose $D \in \mathcal{L}(\mathcal{P}_3(\mathbf{R}), \mathcal{P}_2(\mathbf{R}))$ is the differentiation map defined by $Dp = p'$.

With respect to the basis $1, x, x^2, x^3$ of $\mathcal{P}_3(\mathbb{R})$ and the basis $1, x, x^2$ of $\mathcal{P}_2(\mathbb{R})$, we have

$M(T) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$ since $D1 = 0 = 0(1) + 0(x) + 0(x^2)$

$Dx = 1 = (1)(1) + 0(x) + 0(x^2)$

$Dx^2 = 2x = 0(1) + 2(x) + 0(x^2)$

$Dx^3 = 3x^2 = 0(1) + 0(x) + 3(x^2)$

Addition and Scalar Multiplication of Matrices

3.34 definition: *matrix addition*

The *sum of two matrices of the same size* is the matrix obtained by adding corresponding entries in the matrices:

$$\begin{pmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & & \vdots \\ A_{m,1} & \cdots & A_{m,n} \end{pmatrix} + \begin{pmatrix} C_{1,1} & \cdots & C_{1,n} \\ \vdots & & \vdots \\ C_{m,1} & \cdots & C_{m,n} \end{pmatrix}$$

$$= \begin{pmatrix} A_{1,1} + C_{1,1} & \cdots & A_{1,n} + C_{1,n} \\ \vdots & & \vdots \\ A_{m,1} + C_{m,1} & \cdots & A_{m,n} + C_{m,n} \end{pmatrix}.$$

$$\mathcal{M}(T)v_k = \mathcal{M}(T)_{\cdot,k} \begin{bmatrix} w_1 \\ \vdots \\ w_m \end{bmatrix}$$

$$\mathcal{M}(S)v_k = \mathcal{M}(S)_{\cdot,k} \begin{bmatrix} w_1 \\ \vdots \\ w_m \end{bmatrix}$$

$$[\mathcal{M}(S) + \mathcal{M}(T)]v_k = (\mathcal{M}(S)_{\cdot,k} + \mathcal{M}(T)_{\cdot,k}) \begin{bmatrix} w \\ \vdots \\ w \end{bmatrix}$$

$$\shortparallel$$

$$\mathcal{M}(S+T)v_k$$

3.35 *matrix of the sum of linear maps*

Suppose $S, T \in \mathcal{L}(V, W)$. Then $\mathcal{M}(S + T) = \mathcal{M}(S) + \mathcal{M}(T)$.

$\boxed{\text{Proof}}$ Since $\mathcal{M}(S)$ & $\mathcal{M}(T)$ tell you the coefficients needs to write the image of a basis element in V as a liner combo of a basis of W, add together these coefficients and you, by linearity, get a liner combo that is equal to the sum of the map's image of that basis element.

3.36 definition: *scalar multiplication of a matrix*

The product of a scalar and a matrix is the matrix obtained by multiplying each entry in the matrix by the scalar:

$$\lambda \begin{pmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & & \vdots \\ A_{m,1} & \cdots & A_{m,n} \end{pmatrix} = \begin{pmatrix} \lambda A_{1,1} & \cdots & \lambda A_{1,n} \\ \vdots & & \vdots \\ \lambda A_{m,1} & \cdots & \lambda A_{m,n} \end{pmatrix}.$$

3.38 *the matrix of a scalar times a linear map*

Suppose $\lambda \in \mathbf{F}$ and $T \in \mathcal{L}(V, W)$. Then $\mathcal{M}(\lambda T) = \lambda \mathcal{M}(T)$.

$\boxed{\text{Proof}}$ exercise.

3.39 notation: $\mathbf{F}^{m,n}$

For m and n positive integers, the set of all m-by-n matrices with entries in $\mathbf{F}$ is denoted by $\mathbf{F}^{m,n}$.

3.40 $\dim \mathbf{F}^{m,n} = mn$

Suppose m and n are positive integers. With addition and scalar multiplication defined as above, $\mathbf{F}^{m,n}$ is a vector space of dimension mn.

$\boxed{\text{Proof}}$ That $\mathbb{F}^{m,n}$ is a vector space will be left as an exercise (idea: write a matrix as a long list. Then, everything works since $\mathbb{F}^k$ is a vector space for all k a positive integer). To show $\dim \mathbb{F}^{m,n} = mn$, notice that the list of matrices $B_{1,1}, \dots, B_{1,n}, \dots, B_{m,1}, \dots, B_{m,n}$, where B_{ij} is the matrix w/ zeros everywhere except the ij^{th} entry, is a basis for $\mathbb{F}^{m,n}$. So, since this list has length mn, $\dim \mathbb{F}^{m,n} = mn$.

Matrix Multiplication

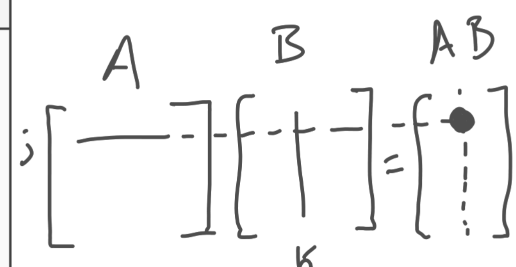
Consider linear maps $T: U \to V$ and $S: V \to W$. The composition ST is a linear map from U to W. Does $\mathcal{M}(ST)$ equal $\mathcal{M}(S)\mathcal{M}(T)$? This question does not yet make sense because we have not defined the product of two matrices. We will choose a definition of matrix multiplication that forces this question to have a positive answer. Let's see how to do this.

3.41 definition: *matrix multiplication*

Suppose A is an m-by-n matrix and B is an n-by-p matrix. Then AB is defined to be the m-by-p matrix whose entry in row j, column k, is given by the equation

$$(AB)_{j,k} = \sum_{r=1}^{n} A_{j,r}B_{r,k}.$$

Thus the entry in row j, column k, of AB is computed by taking row j of A and column k of B, multiplying together corresponding entries, and then summing.



$\boxed{\text{Ex}}$ $\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} 6 & 5 & 4 & 3 \\ 2 & 1 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 10 & 7 & 4 & 1 \\ 26 & 19 & 12 & 5 \\ 42 & 31 & 20 & 9 \end{bmatrix}$

3.43 *matrix of product of linear maps*

If $T \in \mathcal{L}(U, V)$ and $S \in \mathcal{L}(V, W)$, then $\mathcal{M}(ST) = \mathcal{M}(S)\mathcal{M}(T)$.

$\boxed{\text{proof}}$ Let $T \in \mathcal{L}(U,V)$, $S \in \mathcal{L}(V,W)$. Let $v_1, \ldots, v_n$ be a basis of V, $w_1, \ldots, w_m$ be a basis of W, $u_1, \ldots, u_p$ be a basis for U.

Write $A = \mathcal{M}(S)$ & $B = \mathcal{M}(T)$. Now, for $k = 1, \ldots, p$,

$$
\begin{aligned}
(ST)u_k &= S(Tu_k) \\
&= S\left(\sum_{r=1}^{n} B_{r,k} v_r\right) \\
&= \sum_{r=1}^{n} B_{r,k} S(v_r) \quad \leftarrow \text{linearity} \\
&= \sum_{r=1}^{n} B_{r,k} \left(\sum_{j=1}^{m} A_{j,r} w_j\right) \\
&= \sum_{r=1}^{n} \sum_{j=1}^{m} \left(A_{j,r} B_{r,k} w_j\right) \leftarrow \text{distributivity} \\
&= \sum_{j=1}^{n} \left(\sum_{r=1}^{m} A_{j,r} B_{r,k}\right) w_j \leftarrow \text{commutativity}
\end{aligned}
$$

But this says that the k^{th} column of $\mathcal{M}(ST)$ has as its j^{th} entry $\sum_{r=1}^{m} A_{j,r} B_{r,k}$. But, this is just what we get from looking at the j^{th} entry of the k^{th} column of $\mathcal{M}(S)\mathcal{M}(T)$. So, $\mathcal{M}(ST) = \mathcal{M}(S)\mathcal{M}(T)$.