

3B Null Spaces and Ranges

Null Space and Injectivity

3.11 definition: null space, null T

For $T \in \mathcal{L}(V, W)$, the null space of T , denoted by null T , is the subset of V consisting of those vectors that T maps to 0 :

$$\text{null } T = \{v \in V : Tv = 0\}.$$

3.13 the null space is a subspace

Suppose $T \in \mathcal{L}(V, W)$. Then null T is a subspace of V .

Proof Since T is linear, $T(0) = 0$ (we proved this last time). So, $0 \in \text{null } T$.
Now, if $u, v \in \text{null } T$, then note that, by linearity, $T(u+v) = Tu + Tv = 0 + 0 = 0$. So, $u+v \in \text{null } T$ as well.
Finally, if $v \in \text{null } T$ & $\lambda \in \mathbb{F}$, then by linearity, we have $T(\lambda v) = \lambda Tv = \lambda 0 = 0$. So, null T is a subspace.

3.14 definition: injective

A function $T: V \rightarrow W$ is called injective if $Tu = Tv$ implies $u = v$.

3.15 injectivity $\iff$ null space equals $\{0\}$

Let $T \in \mathcal{L}(V, W)$. Then T is injective if and only if null $T = \{0\}$.

Proof First, $\$ T \in \mathcal{L}(V, W)$ is injective. Then, since $T0 = 0$, null $T = \{0\}$ since anything else would imply $Tv = 0 = T0$ for some $v \neq 0$, contradicting injectivity.
Now, $\$ T \in \mathcal{L}(V, W)$ & null $T = \{0\}$. $\$ u, v \in V$ & $Tu = Tv$.
Then, by linearity, we have $0 = Tu - Tv = T(u-v)$. So, $u-v \in \text{null } T$.
But, since null $T = \{0\}$, this says $u-v = 0$, i.e. $u = v$. So, T is injective.

Range and Surjectivity

3.16 definition: *range*

For $T \in \mathcal{L}(V, W)$, the range of T is the subset of W consisting of those vectors that are equal to Tv for some $v \in V$:

$$\text{range } T = \{Tv : v \in V\}.$$

3.18 the range is a subspace

If $T \in \mathcal{L}(V, W)$, then $\text{range } T$ is a subspace of W .

Proof

Let $T \in \mathcal{L}(V, W)$. First, note that $0 = T0$, so $0 \in \text{range } T$. Now, $\forall w_1, w_2 \in \text{range } T$. Then, $\exists v_1, v_2 \in V$ st. $Tv_1 = w_1$ & $Tv_2 = w_2$. Then, by linearity, $T(v_1 + v_2) = Tv_1 + Tv_2 = w_1 + w_2$. So, $w_1 + w_2 \in \text{range } T$ as well. Finally, $\forall w \in \text{range } T$ & $\lambda \in \mathbb{F}$. Then, $\exists v \in V$ st. $Tv = w$ and, by linearity, we have $T(\lambda v) = \lambda Tv = \lambda w$. So, $\lambda w \in \text{range } T$ as well.

3.19 definition: *surjective*

A function $T: V \rightarrow W$ is called *surjective* if its range equals W .

3.20 example: *surjectivity depends on the target space*

The differentiation map $D \in \mathcal{L}(\mathcal{P}_5(\mathbb{R}))$ defined by $Dp = p'$ is not surjective, because the polynomial x^5 is not in the range of D . However, the differentiation map $S \in \mathcal{L}(\mathcal{P}_5(\mathbb{R}), \mathcal{P}_4(\mathbb{R}))$ defined by $Sp = p'$ is surjective, because its range equals $\mathcal{P}_4(\mathbb{R})$, which is the vector space into which S maps.

Fundamental Theorem of Linear Maps

3.21 fundamental theorem of linear maps

Suppose V is finite-dimensional and $T \in \mathcal{L}(V, W)$. Then range T is finite-dimensional and

$$\dim V = \dim \text{null } T + \dim \text{range } T.$$

Proof Since $\text{null } T \leq V$, it is finite dimensional since V is.

Let $u_1, \dots, u_m$ be a basis of $\text{null } T$. Since this is a lin. indep. list in V , it can be extended to a basis of V . Let $u_1, \dots, u_m, v_1, \dots, v_n$ be this extension. Then, $\dim V = m + n = \dim \text{null } T + n$. We will show that $\dim \text{range } T = n$ by showing that $Tv_1, \dots, Tv_n$ is a basis for $\text{range } T$. Let $v \in V$. Since $u_1, \dots, u_m, v_1, \dots, v_n$ is a basis of V , $\exists a_1, \dots, a_m, b_1, \dots, b_n \in \mathbb{F}$ st. $v = a_1 u_1 + \dots + a_m u_m + b_1 v_1 + \dots + b_n v_n$. Then, by linearity, we have $Tv = a_1 Tu_1 + \dots + a_m Tu_m + b_1 Tv_1 + \dots + b_n Tv_n$. Since $u_1, \dots, u_m \in \text{null } T$, $Tu_i = 0$ for $i = 1, \dots, m$. So, $\underbrace{Tv}_{\in \text{range } T} = b_1 Tv_1 + \dots + b_n Tv_n$. Hence, $Tv_1, \dots, Tv_n$ spans $\text{range } T$. Now, we show $Tv_1, \dots, Tv_n$ is lin. indep.: Let $c_1, \dots, c_n \in \mathbb{F}$ be st. $c_1 Tv_1 + \dots + c_n Tv_n = 0$. Then, by linearity, this says $T(c_1 v_1 + \dots + c_n v_n) = 0$. So, $c_1 v_1 + \dots + c_n v_n \in \text{null } T$, and $\exists d_1, \dots, d_m \in \mathbb{F}$ st. $c_1 v_1 + \dots + c_n v_n = d_1 u_1 + \dots + d_m u_m$ (since $u_1, \dots, u_m$ is a basis for $\text{null } T$). But all the c 's & d 's must then be 0 since $u_1, \dots, u_m, v_1, \dots, v_n$ is a basis for V & is lin. indep. So, $Tv_1, \dots, Tv_n$ is a lin. indep. list of vectors that spans $\text{range } T$, i.e., it is a basis of $\text{range } T$.

3.22 linear map to a lower-dimensional space is not injective

Suppose $\underline{V}$ and $\underline{W}$ are finite-dimensional vector spaces such that $\underline{\dim V} > \underline{\dim W}$. Then no linear map from V to W is injective.

Proof let $T \in \mathcal{L}(V, W)$. Then,

$$\dim \text{null } T = \dim V - \dim \text{range } T$$

(since $\dim V = \dim \text{null } T + \dim \text{range } T$ by the FTOLM.)

$$\begin{aligned} \text{But, then, } \dim \text{null } T &= \dim V - \dim \text{range } T \\ &\geq \dim V - \dim W \\ &> 0 \end{aligned}$$

So, $\dim \text{null } T \neq 0 \quad \& \quad \text{therefore } \text{null } T \neq \{0\}.$

So, T is not injective by a previous result.

3.24 linear map to a higher-dimensional space is not surjective

Suppose $\underline{V}$ and $\underline{W}$ are finite-dimensional vector spaces such that $\underline{\dim V} < \underline{\dim W}$. Then no linear map from V to W is surjective.

Proof let $T \in \mathcal{L}(V, W)$. Then, by the FTOLM, we

$$\begin{aligned} \text{have } \dim \text{range } T &= \dim V - \dim \text{null } T \\ &\leq \dim V \\ &< \dim W \end{aligned}$$

So, $\text{range } T \neq W \quad \& \quad T$ cannot be surjective.

Fix n, m positive integers & let $A_{jk} \in \mathbb{F}$ for $j=1, \dots, m$ & $k=1, \dots, n$. Consider the system

$$\left. \begin{array}{l} A_{11}x_1 + \dots + A_{1n}x_n = 0 \\ \vdots \\ A_{m1}x_1 + \dots + A_{mn}x_n = 0 \end{array} \right\} \text{ rhs}$$

Clearly, $x_1 = \dots = x_n = 0$ is a solution. Does any other solution exist?

Define $T: \mathbb{F}^n \rightarrow \mathbb{F}^m$ by

$$T(x_1, \dots, x_n) = (A_{11}x_1 + \dots + A_{1n}x_n, \dots, A_{m1}x_1 + \dots + A_{mn}x_n)$$

3.26 homogeneous system of linear equations

A homogeneous system of linear equations with more variables than equations has nonzero solutions.

Proof The map T above is linear, i.e. $T \in \mathcal{L}(\mathbb{F}^n, \mathbb{F}^m)$

(exercise: check this!). So, if $n > m$, we have

$n = \dim \mathbb{F}^n > \dim \mathbb{F}^m = m$ & T cannot be injective, i.e. $\text{null } T \neq \{0\}$. So, nontrivial solutions exist.

3.28 inhomogeneous system of linear equations

An inhomogeneous system of linear equations with more equations than variables has no solution for some choice of the constant terms.

Proof The inhomogeneous systems w/ solutions are those ones whose rhs lies in $\text{range } T$. But, if $m > n$, this cannot be possible for any $w \in \mathbb{F}^m$ since T cannot be surjective.