3A Vector Space of Linear Maps

vector spaces over the same field

3.1 definition: *linear map*

A *linear map* from V to W is a function $T \colon V \to W$ with the following properties.

additivity

W's +

$T(u + v) = Tu + Tv$ for all $u, v \in V$.

V's + — Different +'s!

homogeneity

$T(\lambda v) = \lambda(Tv)$ for all $\lambda \in \mathbf{F}$ and all $v \in V$.

Different scalings!

3.2 notation: $\mathcal{L}(V, W)$, $\mathcal{L}(V)$

- The set of linear maps from V to W is denoted by $\mathcal{L}(V, W)$.
- The set of linear maps from V to V is denoted by $\mathcal{L}(V)$. In other words, $\mathcal{L}(V) = \mathcal{L}(V, V)$.

$\boxed{\text{Ex}}$

$T : V \to W$ — zero in

- Let $Tv = 0$ for all $v \in V$ i.e. T is the zero map, which is usually also denoted by 0.

Then, $T(v+w) = 0 = 0 + 0 = Tv + Tw$.

similarly, $T(\lambda v) = 0 = \lambda 0 = \lambda(Tv)$.

So, the zero map is a linear map.

- Let $I : V \to V$ be defined by $Iv = v$ for $\forall v \in V$.

Then, $I(v+w) = v + w = Iv + Iw$ & $I(\lambda v) = \lambda v = \lambda(Iv)$.

So, this map (which is called the identity map) is linear too.

- Define $D \in \mathcal{L}(\mathcal{P}(\mathbb{R}))$ by $Dp = p'$. Then, since the derivative is linear, so is D (i.e. $D(p+q) = p' + q'$ & $D(\lambda p) = \lambda p'$)
- Integration also is linear: Define, for example, $T \in \mathcal{L}(\mathcal{P}(\mathbb{R}), \mathbb{R})$ by $Tp = \int_0^1 p(x)\,dx$. Then, $T(p+q) = \int_0^1 p(x) + q(x)\,dx = \int_0^1 p(x)\,dx + \int_0^1 q(x)\,dx = Tp + Tq$ & $T(\lambda p) = \int_0^1 \lambda p(x)\,dx = \lambda \int_0^1 p(x)\,dx = \lambda Tp$

3.4 *linear map lemma*

V finite-dim

Suppose $v_1, \dots, v_n$ is a basis of V and $w_1, \dots, w_n \in W$. Then there exists a unique linear map $T\colon V \to W$ such that

!

$$\boxed{Tv_k = w_k}$$

for each $k = 1, \dots, n$.

$\boxed{\text{Proof}}$ Let V be a finite-dim. V.S. & $v_1, \dots, v_n$ is a basis for V. Let W be a vector space & let $w_1, \dots, w_n \in W$. Define $T: V \to W$

by $\underline{Tv = T(a_1v_1 + \dots + a_nv_n) = a_1w_1 + \dots + a_nw_n}$

(which is possible since $v_1, \dots, v_n$ is a basis & therefore every $v \in V$ can be written uniquely as $v = a_1v_1 + \dots + a_nv_n$ for some $a_1, \dots, a_n \in \mathbb{F}$).

Taking $a_k = 1$ & $a_i = 0$ for $i \neq k$, we get $Tv_k = w_k$.

To see $T \in \mathcal{L}(V, W)$, note that

$$
\begin{aligned}
T(v+w) &= T\big(\underbrace{(a_1v_1 + \dots + a_nv_n)}_{v} + \underbrace{(b_1v_1 + \dots + b_nv_n)}_{w}\big) \\
&= T\big((a_1+b_1)v_1 + \dots + (a_n+b_n)v_n\big) \\
&= (a_1+b_1)w_1 + \dots + (a_n+b_n)w_n \\
&= (a_1w_1 + \dots + a_nw_n) + (b_1w_1 + \dots + b_nw_n) \\
&= T(a_1v_1 + \dots + a_nv_n) + T(b_1v_1 + \dots + b_nv_n) \\
&= Tv + Tw
\end{aligned}
$$

$$
\begin{aligned}
T(\lambda v) &= T\big(\lambda(a_1v_1 + \dots + a_nv_n)\big) \\
&= T\big((\lambda a_1)v_1 + \dots + (\lambda a_n)v_n\big) \\
&= (\lambda a_1)w_1 + \dots + (\lambda a_n)w_n \\
&= \lambda(a_1w_1 + \dots + a_nw_n) \\
&= \lambda T(a_1v_1 + \dots + a_nv_n) \\
&= \lambda Tv
\end{aligned}
$$

So, T is liner. For uniqueness, $\exists$ $T' \in \mathcal{L}(V, W)$ satisfies $T'v_k = w_k$.

Then, $T'(v) = T'(a_1v_1 + \dots + a_nv_n) = a_1(T'v_1) + \dots + a_n(T'v_n) = a_1w_1 + \dots + a_nw_n = T(v)$.

So, $T' = T$ since they agree at every $v \in V$.

Algebraic Operations on $\mathcal{L}(V, W)$

linearity:

$(S+T)(v+w) = S(v+w)+T(v+w)$
$= Sv + Sw + Tv + Tw$
$= Sv + Tv + Sw + Tw$
$= (S+T)(v) + (S+T)(w)$

$(S+T)(\lambda v) = S(\lambda v) + T(\lambda v)$
$= \lambda(Sv) + \lambda(Tv)$
$= \lambda(Sv + Tv)$
$= \lambda(S+T)(v)$

3.5 definition: *addition and scalar multiplication on $\mathcal{L}(V, W)$*

Suppose $S, T \in \mathcal{L}(V, W)$ and $\lambda \in \mathbf{F}$. The *sum* $S + T$ and the *product* λT are the linear maps from V to W defined by

$$(S + T)(v) = Sv + Tv \quad \text{and} \quad (\lambda T)(v) = \lambda(Tv)$$

ex

for all $v \in V$.

3.6 *$\mathcal{L}(V, W)$ is a vector space*

With the operations of addition and scalar multiplication as defined above, $\mathcal{L}(V, W)$ is a vector space.

Proof Routine — leave as an exercise!

3.7 definition: *product of linear maps*

If $T \in \mathcal{L}(U, V)$ and $S \in \mathcal{L}(V, W)$, then the *product* $ST \in \mathcal{L}(U, W)$ is defined by

$$\boxed{(ST)(u) = S(Tu)}$$

for all $u \in U$.

$$U \xrightarrow{T} V \xrightarrow{S} W, \quad U \xrightarrow{ST} W$$

3.8 *algebraic properties of products of linear maps*

$$U \xrightarrow{T_3} V \xrightarrow{T_2} W \xrightarrow{T_1} Y$$

associativity

$(T_1T_2)T_3 = T_1(T_2T_3)$ whenever T_1, T_2, and T_3 are linear maps such that the products make sense (meaning T_3 maps into the domain of T_2, and T_2 maps into the domain of T_1).

identity

$TI = IT = T$ whenever $T \in \mathcal{L}(V, W)$; here the first I is the identity operator on V, and the second I is the identity operator on W.

distributive properties

$(S_1 + S_2)T = S_1T + S_2T$ and $S(T_1 + T_2) = ST_1 + ST_2$ whenever $T, T_1, T_2 \in \mathcal{L}(U, V)$ and $S, S_1, S_2 \in \mathcal{L}(V, W)$.

Proof exercise - routine

3.10 *linear maps take* 0 *to* 0

Suppose T is a linear map from V to W. Then $T(0) = 0$.

$\boxed{\text{Proof}}$ S $T \in \mathcal{L}(V, W)$. Then, we have

$$T(0) = T(0+0) \quad \text{(since 0 is the additive id. of } V)$$

$$T(0) = T(0) + T(0) \quad \text{(by linearity of } T)$$

$$-T(0) + T(0) = -T(0) + (T(0) + T(0)) \quad \text{(add } -T(0) \text{ by existence of additive inverses in } W)$$

$$0 = 0 + T(0) \quad \text{(def'n of additive inverses \& associativity of + in } W)$$

$$0 = T(0) \quad \text{(def'n of additive identity)}$$