2C Dimension

Recall:

2.22 *length of linearly independent list ≤ length of spanning list*

In a finite-dimensional vector space, the length of every linearly independent list of vectors is less than or equal to the length of every spanning list of vectors.

Since a basis is a lin. indep. spanning list, this applies.

2.34 *basis length does not depend on basis*

Any two bases of a finite-dimensional vector space have the same length.

$\boxed{\text{Proof}}$ Let $v_1, \dots, v_n$ and $w_1, \dots, w_m$ be two bases for V. By the result above, $n \leq m$ since $v_1, \dots, v_n$ is lin indep. & $w_1, \dots, w_m$ spans V. Now, notice that $w_1, \dots, w_m$ is also lin. indep. & $v_1, \dots, v_n$ is also a spanning list. Hence, we also have $m \leq n$. Therefore, $m = n$.

This only works b/c every basis has the same length!

2.35 definition: *dimension*, $\dim V$

- The *dimension* of a finite-dimensional vector space is the length of any basis of the vector space.
- The dimension of a finite-dimensional vector space V is denoted by $\dim V$.

i.e. $\dim : \underset{\text{Vector spaces}}{\text{finite dim}} \to \mathbb{N}$

where $\mathbb{N} = \{0, 1, 2, \dots\}$

2.37 *dimension of a subspace*

If V is finite-dimensional and U is a subspace of V, then $\dim U \leq \dim V$.

$\boxed{\text{Proof}}$ Let $U \leq V$, where V is finite dim.
Let $u_1, \dots, u_n$ be a basis for U (i.e. $\dim U = n$) & let $v_1, \dots, v_m$ be a basis for V. Then, notice that $u_1, \dots, u_n$ is a linearly indep. list in V as well, and $v_1, \dots, v_m$ is a spanning list in V. By the result we recalled at the beginning of lecture, $n \leq m$.
That is, $\dim U \leq \dim V$.

2.38 *linearly independent list of the right length is a basis*

Suppose V is finite-dimensional. Then every linearly independent list of vectors in V of length $\dim V$ is a basis of V.

$\boxed{\text{Proof}}$ Suppose V is finite dim. & $v_1, \dots, v_n$ is a lin. indep. list in V. Suppose further that $\dim V = n$. Recall that any lin. indep. list in a finite dim. V.S. can be extended to a basis. But, since the length of every basis is the same in a finite dim V.S., this extention must be trivial (i.e. it must not add anything!). Hence, $v_1, \dots, v_n$ is already a basis whenever it is linearly indep. & $\dim V = n$.

2.39 *subspace of full dimension equals the whole space*

Suppose that V is finite-dimensional and U is a subspace of V such that $\dim U = \dim V$. Then $U = V$.

$\boxed{\text{Proof}}$ Let V be finite dim & $U \leq V$ & s.t. $\dim U = \dim V$. Let $u_1, \ldots, u_n$ be a basis for U. Then, $u_1, \ldots, u_n$ is a linearly indep. list in V & its length is $\dim V$ since $\dim U = \dim U$. But, by the previous result, any lin. indep. list of length $\dim V$ is a basis. So, $u_1, \ldots, u_n$ is a basis for V as well; i.e., $U = \operatorname{Span}(u_1, \ldots, u_n) = V$.

2.42 *spanning list of the right length is a basis*

Suppose V is finite-dimensional. Then every spanning list of vectors in V of length $\dim V$ is a basis of V.

$\boxed{\text{Proof}}$ S.t. V is finite dim. & let $v_1, \ldots, v_n$ be a spanning list of length $\dim V = n$. Since every spanning list in a finite dim. V.S. can be reduced to a basis, $v_1, \ldots, v_n$ can as well. But, since every basis has the same length, this reduction must be trivial (i.e., we remove nothing!). So, $v_1, \ldots, v_n$ is already a basis for V.

2.43 *dimension of a sum*

If V_1 and V_2 are subspaces of a finite-dimensional vector space, then

$$\dim(V_1 + V_2) = \dim V_1 + \dim V_2 - \dim(V_1 \cap V_2).$$

$\boxed{\text{Proof}}$ Let $V_1, V_2 \subseteq V$, a finite dim. V.S..

Let $v_1, \ldots, v_m$ be a basis for $V_1 \cap V_2$ so that $\dim V_1 \cap V_2 = m$. Since $v_1, \ldots, v_m$ is a linearly indp. spanning list in $V_1 \cap V_2$, it is a lin. indp list in both V_1 \$ V_2, so we can extend it in two ways:

$v_1, \ldots, v_m, u_1, \ldots, u_j$ a basis of V_1

$v_1, \ldots, v_m, w_1, \ldots, w_k$ a basis of V_2

Then, we are saying that $\dim V_1 = m + j$ \$ $\dim V_2 = m + k$

We will show that $B = v_1, \ldots, v_m, u_1, \ldots, u_j, w_1, \ldots, w_k$ is a basis for $V_1 + V_2$, since this says that:

$$\begin{aligned} \dim V_1 + V_2 = m + j + k &= (m+j) + (m+k) - m \\ &= \dim V_1 + \dim V_2 - \dim V_1 \cap V_2. \end{aligned}$$

Notice that B is contained in $V_1 \cup V_2$ \$ is therefore contained in $V_1 + V_2$. The span of B certainly contains both V_1 \$ V_2, so it is equal to $V_1 + V_2$ (i.e. $\operatorname{Span} B \subseteq V_1 + V_2$ \$ $V_1 + V_2 \subseteq \operatorname{Span} B$, so $\operatorname{span} B = V_1 + V_2$). Hence, B spans $V_1 + V_2$.

Now, we need only show linear indp.

$\boxed{\text{Proof}}$ (continued)

To show lin. independence, suppose that

$$a_1 v_1 + \dots + a_m v_m + b_1 u_1 + \dots + b_j u_j + c_1 w_1 + \dots + c_k w_k = 0$$

for some $a_1, \dots, a_m, b_1, \dots, b_j, c_1, \dots, c_k \in \mathbb{F}$.

Notice that this can be rewritten as $c_1 w_1 + \dots + c_k w_k = -(a_1 v_1 + \dots + a_m v_m + b_1 u_1 + \dots + b_j u_j)$

Hence, $c_1 w_1 + \dots + c_k w_k \in V_1$, since $v_1, \dots, v_m, u_1, \dots, u_j$ is a basis for V_1.

All of $w_1, \dots, w_k$ belong to V_2 since $v_1, \dots, v_m, w_1, \dots, w_k$ is a basis for V_2.

So, $c_1 w_1 + \dots + c_k w_k \in V_1 \cap V_2$. Since $v_1, \dots, v_m$ is a basis for $V_1 \cap V_2$,

$c_1 w_1 + \dots + c_k w_k = d_1 v_1 + \dots + d_m v_m$ for some $d_1, \dots, d_m \in \mathbb{F}$. However, $v_1, \dots, v_m, w_1, \dots, w_k$ is lin. indep. since it is a basis for V_2, so this implies that

$c_1 = \dots = c_k = d_1 = \dots = d_m = 0$, since it implies $c_1 w_1 + \dots + c_k w_k - d_1 v_1 - \dots - d_m v_m = 0$.

So, we can rewrite it as $a_1 v_1 + \dots + a_m v_m + b_1 u_1 + \dots b_j u_j = 0$. Now, repeat for the list $v_1, \dots v_m, u_1, \dots, u_j$, which is also lin. indep. since it is a basis for V_1. Hence, $a_1 = \dots = a_m = b_1 = \dots = b_j = 0$. So, B is lin. indep. and, since it spans $V_1 + V_2$, it is a basis of $V_1 + V_2$.

sets	**vector spaces**
S is a finite set	V is a finite-dimensional vector space
$\#S$	$\dim V$
for subsets S_1, S_2 of S, the union $S_1 \cup S_2$ is the smallest subset of S containing S_1 and S_2	for subspaces V_1, V_2 of V, the sum $V_1 + V_2$ is the smallest subspace of V containing V_1 and V_2
$\#(S_1 \cup S_2) = \#S_1 + \#S_2 - \#(S_1 \cap S_2)$	$\dim(V_1 + V_2) = \dim V_1 + \dim V_2 - \dim(V_1 \cap V_2)$
$\#(S_1 \cup S_2) = \#S_1 + \#S_2 \iff S_1 \cap S_2 = \emptyset$	$\dim(V_1 + V_2) = \dim V_1 + \dim V_2 \iff V_1 \cap V_2 = \{0\}$
$S_1 \cup \dots \cup S_m$ is a disjoint union $\iff$ $\#(S_1 \cup \dots \cup S_m) = \#S_1 + \dots + \#S_m$	$V_1 + \dots + V_m$ is a direct sum $\iff$ $\dim(V_1 + \dots + V_m) = \dim V_1 + \dots + \dim V_m$

Basically, the same thing!
This is the power of dimension!