2B Bases

tells us when the span doesn't have any extraneous info

2.26 definition: *basis*

A *basis* of V is a list of vectors in V that is linearly independent and spans V.

How to make subspaces from lists of vectors

$\boxed{\text{Ex}}$ $v_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

Then, if $v = \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2$, notice that $a_1v_1 + a_2v_2 = \begin{bmatrix} 0 \\ a_1 \end{bmatrix} + \begin{bmatrix} 2a_2 \\ 0 \end{bmatrix} = \begin{bmatrix} 2a_2 \\ a_1 \end{bmatrix}$

& $v = a_1v_1 + a_2v_2$ has the unique solution $a_2 = \frac{x}{2}, a_1 = y$.

So, v_1, v_2 is a basis of $\mathbb{R}^2$.

2.28 *criterion for basis*

A list $v_1, \ldots, v_n$ of vectors in V is a basis of V if and only if every $v \in V$ can be written uniquely in the form

2.29
$$v = a_1v_1 + \cdots + a_nv_n,$$

where $a_1, \ldots, a_n \in \mathbf{F}$.

$\boxed{\text{Proof}}$ & $v_1, \ldots, v_n$ is a basis of V.

Let $v \in V$. Since $v_1, \ldots, v_n$ spans V, it follows that there are scalars $a_1, \ldots, a_n \in \mathbb{F}$ s.t. $v = a_1v_1 + \cdots + a_nv_n$. Now, & that $c_1, \ldots, c_n \in \mathbb{F}$ also satisfy $v = c_1v_1 + \cdots + c_nv_n$. Then,

$$0 = v - v = (a_1v_1 + \cdots + a_nv_n) - (c_1v_1 + \cdots + c_nv_n)$$
$$= (a_1 - c_1)v_1 + \cdots + (a_n - c_n)v_n$$

But, this means that $a_i - c_i = 0$ for $i = 1, \ldots, n$.

That is, $a_i = c_i$.

In the reverse, & $v = a_1v_1 + \cdots + a_nv_n$ has a unique solution $a_1, \ldots, a_n \in \mathbb{F}$ for $\forall v \in V$. This certainly implies $v_1, \ldots, v_n$ spans V, since it says every $v \in V$ is a linear combo of $v_1, \ldots, v_n$. For lin. indep., notice that this implies in particular that the solution $a_1 = \cdots = a_n = 0$ to $0 = a_1v_1 + \cdots + a_nv_n$ is unique. So, $v_1, \ldots, v_n$ is also lin. indep. Hence, it is a basis.

2.30 *every spanning list contains a basis*

Every spanning list in a vector space can be reduced to a basis of the vector space.

$\boxed{\text{proof}}$ Let $v_1, \dots, v_n$ be a spanning list for V.

Step 1: If $v_1 = 0$, delete v_1. If not, do nothing.

Step k: If v_k is in $\operatorname{span}(v_1, \dots, v_{k-1})$, delete v_k.
If not, do nothing.

This process eventually stops: Either we discard nothing & have only the list $v_1, \dots, v_n$ which spans V, or we have some new list $w_1, \dots, w_m$ w/ $w_i \in \{v_1, \dots, v_n\}$ & $w_i = v_j \Rightarrow w_t \in \{v_1, \dots, v_j\}$ for $t < i$ (i.e. still in order, just missing a few) and, by the lin. dep. lemma, this list is independent since, if it were dependent, we could choose some w_k s.t. $w_k \in \operatorname{Span}(w_1, \dots, w_{k-1})$ & by construction this is not the case. Hence, we are done!

2.31 *basis of finite-dimensional vector space*

Every finite-dimensional vector space has a basis.

$\boxed{\text{Proof}}$ If V is finite dim., it has a spanning list of vectors. By the previous, this can be reduced to a basis.

2.32 *every linearly independent list extends to a basis*

Every linearly independent list of vectors in a finite-dimensional vector space can be extended to a basis of the vector space.

$\boxed{\text{Proof}}$ Let $\underline{u_1, \ldots, u_m}$ be a lin. indep. list of vectors in V, a finite dimensional vector space. Since V is finite dim, there exists a spanning list for V, call it $\underline{w_1, \ldots, w_n}$. Form the list $\underline{u_1, \ldots, u_m, w_1, \ldots, w_n}$. This certainly spans V and can be reduced to a basis using the process from the proof of the result before last. This process deletes none of the u_i, since $u_i \notin \operatorname{span}(u_1, \ldots, u_{i-1})$ for any $i \in 1, \ldots, m$ as $u_1, \ldots, u_m$ is linearly indep. Hence, the obtained basis "starts w/" the u_i's. So, we are done!

2.33 *every subspace of V is part of a direct sum equal to V*

Suppose V is finite-dimensional and U is a subspace of V. Then there is a subspace W of V such that $V = U \oplus W$.

$\boxed{\text{Proof}}$ Suppose V is finite dim. Let $U \leq V$. Then, U is also finite dim. Hence, there is a basis $u_1, \ldots, u_m$ of U. Notice that $u_1, \ldots, u_m$ is linearly indep. in V, so we can extend it to a basis $u_1, \ldots, u_m, w_1, \ldots, w_n$ of V, by the previous result.

Let $W = \operatorname{span}(w_1, \ldots, w_n)$. Then, since $u_1, \ldots, u_m, w_1, \ldots, w_n$ is a basis for V, every $v \in V$ can be written as

$$v = a_1 u_1 + \cdots + a_m u_m + b_1 w_1 + \cdots + b_n w_n$$

for some $a_1, \ldots, a_m, b_1, \ldots, b_n \in \mathbb{F}$. To get directness, recall that we need only show that $V = U + W$ and $U \cap W = \{0\}$. Since $a_1 u_1 + \cdots + a_m u_m \in U$ for any $a_1, \ldots, a_m \in \mathbb{F}$ and $b_1 w_1 + \cdots + b_n w_n \in W$ for any $b_1, \ldots, b_n \in \mathbb{F}$, the fact here tells us we can write any $v \in V$ as $v = u + w$ for some $u \in U$ and $w \in W$. Hence, $V = U + W$. Now, since $u_1, \ldots, u_m, w_1, \ldots, w_n$ is a basis, it is linearly indep. and the only way to write $0 = a_1 u_1 + \cdots + a_m u_m + b_1 w_1 + \cdots + b_n w_n$ is w/ $a_1 = \cdots = a_m = b_1 = \cdots = b_n = 0$.
So, suppose $v \in U \cap W$. Then, $v = a_1 u_1 + \cdots + a_m u_m$ for some $a_1, \ldots, a_m \in \mathbb{F}$ and $v = b_1 w_1 + \cdots + b_n w_n$ for some $b_1, \ldots, b_n \in \mathbb{F}$. Hence, $0 = v - v = a_1 u_1 + \cdots + a_m u_m - b_1 w_1 - \cdots - b_n w_n$ and we get $a_1 = \cdots = a_m = -b_1 = \cdots = -b_n = 0$ and $v = 0$. So, $V = U \oplus W$.