2.2 definition: *linear combination*

A *linear combination* of a list $v_1, \dots, v_m$ of vectors in V is a vector of the form

$$a_1 v_1 + \cdots + a_m v_m,$$

where $a_1, \dots, a_m \in \mathbf{F}$.

2.4 definition: *span*

The set of all linear combinations of a list of vectors $v_1, \dots, v_m$ in V is called the *span* of $v_1, \dots, v_m$, denoted by $\operatorname{span}(v_1, \dots, v_m)$. In other words,

$$\operatorname{span}(v_1, \dots, v_m) = \{a_1 v_1 + \cdots + a_m v_m : a_1, \dots, a_m \in \mathbf{F}\}.$$

The span of the empty list $(\,)$ is defined to be $\{0\}$. ← convention

2.6 *span is the smallest containing subspace*

The span of a list of vectors in V is the smallest subspace of V containing all vectors in the list.

$\boxed{\text{Proof}}$ Notice: If $v_1, \dots, v_m \in V$ is a list of vectors, then $0v_1 + \dots + 0v_m = 0$, so $0 \in \operatorname{span}(v_1, \dots, v_m)$.

Moreover, if $v \in \operatorname{Span}(v_1, \dots, v_m)$, then $v = a_1 v_1 + \dots + a_m v_m$ for some $a_1, \dots, a_m \in \mathbb{F}$ & for any $b \in \mathbb{F}$, we have

$b\,v = b(a_1 v_1 + \dots + a_m v_m) = (ba_1)v_1 + \dots + (ba_m)v_m \in \operatorname{Span}(v_1, \dots, v_m)$.

Similarly, if $v' = a_1' v_1 + \dots + a_m' v_m \in \operatorname{Span}(v_1, \dots, v_m)$, then

$$\begin{aligned} v + v' &= (a_1 v_1 + \dots + a_m v_m) + (a_1' v_1 + \dots + a_m' v_m) \\ &= (a_1 v_1 + a_1' v_1) + \dots + (a_m v_m + a_m' v_m) \\ &= (a_1 + a_1')v_1 + \dots + (a_m + a_m')v_m \in \operatorname{Span}(v_1, \dots, v_m) \end{aligned}$$

So, by a theorem we proved last time, $\operatorname{Span}(v_1, \dots, v_m)$ is a subspace of V. Moreover, notice that $v_i = 0v_1 + \dots + 0v_{i-1} + (1)v_i + 0v_{i+1} + \dots + 0v_m$, so $v_i \in \operatorname{Span}(v_1, \dots, v_m)$ for $i = 1, \dots, m$.

Span$(v_1, \dots, v_m)$ is a subspace containing $v_1, \dots, v_m$

Now, $\forall$ $W \leq V$ & $v_1, \dots, v_m \in W$. Then, since linear combos of $v_1, \dots, v_m$ are certainly also contained in W, $\operatorname{Span}(v_1, \dots, v_m) \subseteq W$ as well.

It is the smallest one.

2.7 definition: *spans*

If $\operatorname{span}(v_1, \dots, v_m)$ equals V, we say that the list $v_1, \dots, v_m$ *spans* V.

2.9 definition: *finite-dimensional vector space*

A vector space is called *finite-dimensional* if some list of vectors in it spans the space.

Recall that by definition every list has finite length.

2.13 definition: *infinite-dimensional vector space*

A vector space is called *infinite-dimensional* if it is not finite-dimensional.

2.10 definition: *polynomial*, $\mathcal{P}(\mathbf{F})$

- A function $p\colon \mathbf{F} \to \mathbf{F}$ is called a *polynomial* with coefficients in $\mathbf{F}$ if there exist $a_0, \dots, a_m \in \mathbf{F}$ such that

$$p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_m z^m$$

for all $z \in \mathbf{F}$.

- $\mathcal{P}(\mathbf{F})$ is the set of all polynomials with coefficients in $\mathbf{F}$.

2.12 notation: $\mathcal{P}_m(\mathbf{F})$

For m a nonnegative integer, $\mathcal{P}_m(\mathbf{F})$ denotes the set of all polynomials with coefficients in $\mathbf{F}$ and degree at most m.

$\boxed{\text{Ex}}$ $\mathcal{P}_m(\mathbb{F}) \subseteq \mathcal{P}(\mathbb{F})$

i.e. add togther two polys of degree at most m, get back a poly of degree at most m. Similarly, scale a poly. of degree at most m, get back a poly of the same degree or, if the scalar is 0, a lower degree poly.

Notice: $\mathcal{P}_m(\mathbb{F}) = \operatorname{span}(1, z, z^2, \dots, z^m)$

So, $\mathcal{P}_m(\mathbb{F})$ is finite dimensional!

However, $\mathcal{P}(\mathbb{F})$ is not!

2.15 definition: *linearly independent*

- A list $v_1, \ldots, v_m$ of vectors in V is called *linearly independent* if the only choice of $a_1, \ldots, a_m \in \mathbf{F}$ that makes
$$a_1 v_1 + \cdots + a_m v_m = 0$$
is $a_1 = \cdots = a_m = 0$.
- The empty list $(\,)$ is also declared to be linearly independent.

2.17 definition: *linearly dependent*

- A list of vectors in V is called *linearly dependent* if it is not linearly independent.
- In other words, a list $v_1, \ldots, v_m$ of vectors in V is linearly dependent if there exist $a_1, \ldots, a_m \in \mathbf{F}$, not all 0, such that $a_1 v_1 + \cdots + a_m v_m = 0$.

2.19 *linear dependence lemma*

Suppose $v_1, \ldots, v_m$ is a linearly dependent list in V. Then there exists $k \in \{1, 2, \ldots, m\}$ such that
$$v_k \in \operatorname{span}(v_1, \ldots, v_{k-1}).$$
Furthermore, if k satisfies the condition above and the k^{th} term is removed from $v_1, \ldots, v_m$, then the span of the remaining list equals $\operatorname{span}(v_1, \ldots, v_m)$.

$\boxed{\text{Proof}}$ Since $v_1, \ldots, v_m$ is lin. dep., there are scalars $a_1, \ldots, a_m \in \mathbb{F}$, not all zero, such that $a_1 v_1 + \cdots + a_m v_m = 0$. Let k be the largest # s.t. $a_k \neq 0$. Then, $0 = a_1 v_1 + \cdots + a_k v_k + a_{k+1} v_{k+1} + \cdots + a_m v_m$

$$0 = a_1 v_1 + \cdots + a_k v_k + 0 + \cdots + 0$$

So, $-a_k v_k = a_1 v_1 + \cdots + a_{k-1} v_{k-1}$ and, since $a_k \neq 0$, we can multiply by $(a_k)^{-1}$ to get

$$v_k = \underline{(-a_k)^{-1} a_1 v_1 + \cdots + (-a_k)^{-1} a_{k-1} v_{k-1}}$$

Hence, $v_k \in \operatorname{Span}(v_1, \ldots, v_{k-1})$. To see the rest, notice that this lets us rewrite any scaling of v_k as a linear combo of $v_1, \ldots, v_{k-1}$.

2.22 *length of linearly independent list $\leq$ length of spanning list*

In a finite-dimensional vector space, the length of every linearly independent list of vectors is less than or equal to the length of every spanning list of vectors.

$\boxed{\text{Proof}}$ Let $\text{span}(w_1, \ldots, w_n) = V$, i.e. $w_1, \ldots, w_n$ is a spanning list. Let $u_1, \ldots, u_m$ be a linearly indep. list.

Since $w_1, \ldots, w_n$ spans V, so does $u_1, w_1, \ldots, w_n$.

However, since $u_1 \in V$, $u_1, w_1, \ldots, w_n$ is linearly dependent.

So, by the lin. dep. lemma, $\exists k$ s.t. the list

$\underbrace{u_1, w_1, \ldots, w_{k-1}, w_{k+1}, \ldots, w_n}_{\text{length } n}$ spans V.

This is step 1. Here is step k for $k = 2, \ldots, m$:

Take your list from the previous step & notice that u_k is in its span. So, adding u_k in the k^{th} position in the list yields a lin. dep. list & we can use the lin. dep. lemma to remove an element. Since the start of this list is all u_i's & $u_1, \ldots, u_m$ is lin. indep., we get that the removed element must be a w_i.

Then, at the end of this process, we get a list that looks like $\underline{u_1, \ldots, u_m}, w_{t_1}, w_{t_2}, \ldots, w_{t_{n-m}}$ (where there are no w's in the case where $n = m$) that spans V.

So, the length of such a list must be at least as long as the length of the lin. indep. list $u_1, \ldots, u_m$ (i.e. m).

2.25 *finite-dimensional subspaces*

Every subspace of a finite-dimensional vector space is finite-dimensional.

$\boxed{\text{Proof}}$ Let V be a fin. dim. VS. Then,

Let $U \leq V$.

Step 1: If $U = \{0\}$, we are done since $U = \text{span}(\ )$.

If not, let $u_1 \in U$ be non-zero.

Step j: Take $\text{Span}(u_1, \ldots, u_{j-1})$. If this is equal to U,

for $j \in \{2, \ldots, n\}$ we are done. If not, take some $u_j \notin \text{Span}(u_1, \ldots, u_{j-1})$

& repeat now with $\text{Span}(u_1, \ldots, u_j)$ as the next step.

Now, since $u_j \notin \text{Span}(u_1, \ldots, u_{j-1})$, the list $u_1, \ldots, u_j$ is lin. indep. By the previous result, the length of every lin. indep. list in V is less than or equal to the length of any list that spans V. Since V is finite dim., there is a finite length list that spans V, so there is an upper bound on how long a lin. indep. list in U can be.

This applies to the u_i's — the process has to eventually stop at some k that is less than or equal to the length of any choice of a finite spanning list for V. Hence, we get $U = \text{Span}(u_1, \ldots, u_k)$ & U is finite dim.