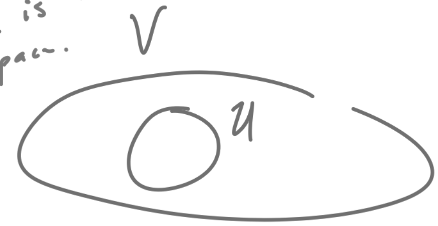
1C Subspaces

$U \subseteq V \leftarrow$ subset

$U \leq V \leftarrow$ subspace.

we write $U \leq V$ to denote that U is a subspace.



1.33 definition: *subspace*

A subset U of V is called a *subspace* of V if U is also a vector space with the same additive identity, addition, and scalar multiplication as on V.

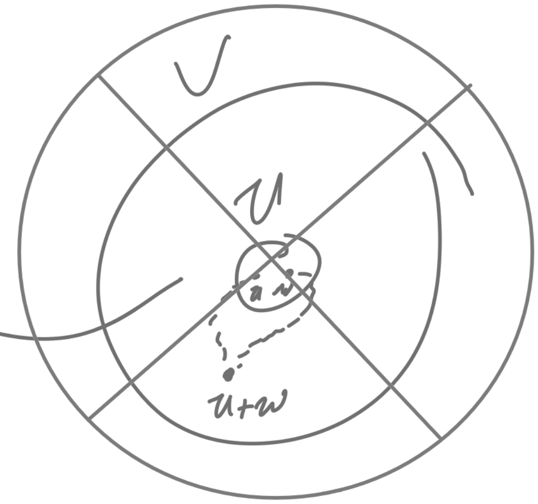
1.34 *conditions for a subspace*

A subset U of V is a subspace of V if and only if U satisfies the following three conditions.

(1) **additive identity**
$0 \in U$.

(2) **closed under addition**
$u, w \in U$ implies $u + w \in U$.

(3) **closed under scalar multiplication**
$a \in \mathbf{F}$ and $u \in U$ implies $au \in U$.



Proof (Sketch)

If U is a subspace, then this holds by definition.
In the reverse, notice that any of these being violated would prevent U from being a subspace. Moreover, they are precisely the 3 conditions defining a subspace.

1.36 definition: *sum of subspaces*

Suppose $V_1, \ldots, V_m$ are subspaces of V. The *sum* of $V_1, \ldots, V_m$, denoted by $V_1 + \cdots + V_m$, is the set of all possible sums of elements of $V_1, \ldots, V_m$. More precisely,

$$V_1 + \cdots + V_m = \{v_1 + \cdots + v_m : v_1 \in V_1, \ldots, v_m \in V_m\}.$$

Ex

$$U = \{(x, x, y, y) \in \mathbb{F}^4 : x, y \in \mathbb{F}\}$$

$$(x_1, x_1, y_1, y_1) + (x_2, x_2, x_2, y_2) = (x_1 + x_2, x_1 + x_2, y_1 + x_2, y_1 + y_2)$$

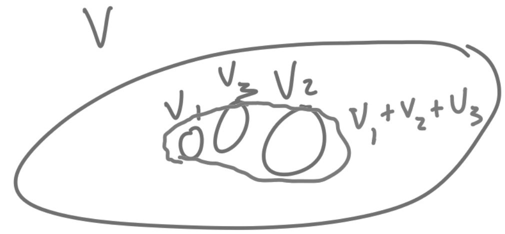
Same

$$W = \{(x, x, x, y) \in \mathbb{F}^4 : x, y \in \mathbb{F}\}$$

$$U + W = \{(x, x, y, z) \in \mathbb{F}^4 : x, y, z \in \mathbb{F}\}$$

1.40 *sum of subspaces is the smallest containing subspace*

Suppose $V_1, \dots, V_m$ are subspaces of V. Then $V_1 + \cdots + V_m$ is the smallest subspace of V containing $V_1, \dots, V_m$.



Proof

Clearly, $V_i \subseteq V_1 + \dots + V_m$, since

$v_i = 0 + \dots + 0 + v_i + 0 + \dots + 0 + 0 \in V_1 + \dots + V_m$ for any $v_i \in V_i$.

Conversely, if $W \leq V$ such that $V_i \subseteq W$ for $i = 1, \dots, m$,

then $V_1 + \dots + V_m \subseteq W$ as well, as subspaces must contain all finite sums of their elements.

1.41 definition: *direct sum,* $\oplus$

Suppose $V_1, \ldots, V_m$ are subspaces of V.

- The sum $V_1 + \cdots + V_m$ is called a *direct sum* if each element of $V_1 + \cdots + V_m$ can be written in only one way as a sum $v_1 + \cdots + v_m$, where each $v_k \in V_k$.
- If $V_1 + \cdots + V_m$ is a direct sum, then $V_1 \oplus \cdots \oplus V_m$ denotes $V_1 + \cdots + V_m$, with the $\oplus$ notation serving as an indication that this is a direct sum.

$\boxed{\text{Ex}}$

Let $0 \neq U \subseteq V$. Then, $U+U$ is not direct! If $w \in U+U$, then notice that $w \in U$, so $w = 0 + w = w + 0$ i.e. $w = u_1 + u_2 = u_1' + u_2'$ for $u_i' \neq u_i$.

1.45 *condition for a direct sum*

Suppose $V_1, \ldots, V_m$ are subspaces of V. Then $V_1 + \cdots + V_m$ is a direct sum if and only if the only way to write 0 as a sum $v_1 + \cdots + v_m$, where each $v_k \in V_k$, is by taking each v_k equal to 0.

$\boxed{\text{Proof}}$ If $V_1 + \cdots + V_m$ is a direct sum, then since $0 = 0 + \cdots + 0$, we get our desired result.

Now, suppose the only way to write $0 \in V_1 + \cdots + V_m$ is as $0 = 0 + \cdots + 0$, i.e., $0 = v_1 + \cdots + v_m$ implies $v_i = 0$ for $i = 1, \ldots, m$.

Now, if $w = v_1 + \cdots + v_m$ & $w = v_1' + \cdots + v_m'$ for some $w \in V_1 + \cdots V_m$.

Then, $0 = w - w = (v_1 + \cdots + v_m) - (v_1' + \cdots + v_m')$

$$= (v_1 - v_1') + \cdots + (v_m - v_m')$$

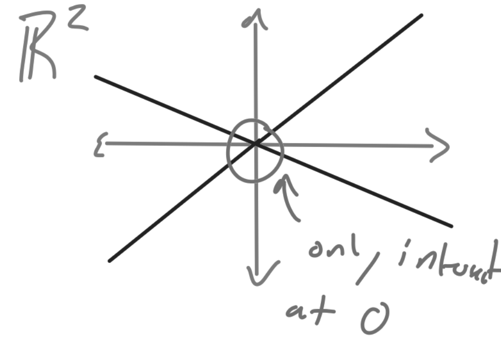
Since $v_i - v_i' \in V_i$ for $i = 1, \ldots, m$, we get $v_i - v_i' = 0$

Hence, $v_i = v_i'$ and there is only one way to write w as a sum of elements of V_i for $i = 1, \ldots, m$. That is, $V_1 + \cdots + V_m$ is direct.

1.46 *direct sum of two subspaces*

Suppose U and W are subspaces of V. Then

$$U + W \text{ is a direct sum} \iff U \cap W = \{0\}.$$



Proof

§ $U+W$ is direct. If $v \in U \cap W$, then $0 = v + (-v)$ since $v \in U \cap W \Rightarrow v \in U$ & $v \in W$ (and, hence, $-v \in W$). So, by the previous result, $v = -v = 0$. Hence, $U \cap W = \{0\}$.

Now, § $\underline{U \cap W = \{0\}}$. Then, if $u \in U$ & $w \in W$ are such that $u + w = 0$, then $u = -w$. Hence, $u \in W$ as well (since W is a vector space). But, only 0 belongs to both W & U. So $w = 0$. Hence, $u + w = 0 \Rightarrow u = w = 0$ & by the previous result, $U+W$ is direct.