Def'n A permutation of $(1,\dots,n)$ is a list $(m_1,\dots,m_n)$ where each of $1,\dots,n$ appears exactly once. The set of all permutations of $(1,\dots,n)$ is denoted S_n.

Def'n The sign of a permutation $(m_1,\dots,m_n)\in S_n$ is the quantity $(-1)^t$ where t is the # of pairs (j,k) w/ $1\leq j\leq k\leq n$ s.t. j appears after k in $(m_1,\dots,m_n)$.

Ex

- $(2,1,3,4)\in S_4$ has sign -1, since only $(2,1)$ is out of order
- $(2,3,\dots,n,1)\in S_n$ has sign $(-1)^{n-1}$

Def'n Suppose A is an $n\times n$ matrix. Then,

$$\text{"det" } A = \sum_{(m_1,\dots,m_n)\in S_n} \operatorname{sign}(m_1,\dots,m_n)\, A_{m_1,1}\cdots A_{m_n,n}.$$

Ex

$$\text{"det"} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \sum_{(m_1,m_2)\in S_2} \operatorname{sign}(m_1,m_2)\, A_{m_1,1} A_{m_2,2}$$

$$= \operatorname{sign}(1,2) A_{1,1} A_{2,2} + \operatorname{sign}(2,1) A_{2,1} A_{1,2}$$

$$= A_{1,1} A_{2,2} - A_{2,1} A_{1,2}$$

Lemma If A is upper triangular. Then, "det" $A = A_{11} \cdots A_{nn}$.

proof The permutation $(1, \ldots, n) \in S_n$ contributes the term $\text{sign}(1, \ldots, n) A_{1,1} \cdots A_{n,n} = A_{1,1} \cdots A_{n,n}$. For any other permutation $(m_1, \ldots, m_n) \in S_n$, there is some entry m_K s.t. $K < m_K$. But, $\underline{A_{m_K, K} = 0}$ in this case since A is upper triangular. So, $\text{sign}(m_1, \ldots, m_n) A_{m_1,1} \cdots A_{m_n,n} = 0$. Therefore,

$$\text{"det" } A = \sum_{(m_1, \ldots, m_n) \in S_n} \text{sign}(m_1, \ldots, m_n) A_{m_1,1} \cdots A_{m_n,n}$$
$$= A_{1,1} \cdots A_{n,n}$$

Lemma "det"(AB) = "det"(BA) for A, B $n \times n$ matrices.

Proof (sketch) Just write it out & use permutations to reorder everything.

Theorem "det"$(\mathcal{M}(T, (u_1, \ldots, u_n)))$ = "det"$(\mathcal{M}(T, (v_1, \ldots, v_n)))$ for $T \in \mathcal{L}(V)$ & $(v_1, \ldots, v_n)$, $(u_1, \ldots, u_n)$ two bases of V.

Proof By the change of basis formula, there is an invertible matrix C s.t. $A = C^{-1} B C$ where $A = \mathcal{M}(T, (u_1, \ldots, u_n))$ & $B = \mathcal{M}(T, (v_1, \ldots, v_n))$.
So, using the last lemma, "det"(A) = "det"$(C^{-1}BC)$ = "det"$(BC^{-1}C)$ = "det"(B).

Theorem "det" = det (i.e. $\det(T)$ = "det"$(\mathcal{M}(T))$)

Proof Use Jordan normal form along w/ our lemma re: upper triangular matrices!

Theorem If V is an inner product space & $S \in \mathcal{L}(V)$ is unitary. Then, $|\det S| = 1$.

Proof $|\det S| = |\lambda_1 \cdots \lambda_n| = |\lambda_1| \cdots |\lambda_n| = 1 \cdots 1 = 1$

where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of S repeated according to their multiplicity.

Theorem If V is an inner product space & $T \in \mathcal{L}(V)$. Then, $\det T = s_1 \cdots s_n$ where $s_1, \ldots, s_n$ are the singular values of T.

Proof By SVD, $\exists$ ONBs $e_1, \ldots, e_n$ & $f_1, \ldots, f_n$ of V s.t. $Tv = s_1 \langle v, e_1 \rangle f_1 + \cdots + s_n \langle v, e_n \rangle f_n$. Take $S \in \mathcal{L}(V)$ so that $Se_k = f_k$ (this defines S since $e_1, \ldots, e_n$ is a basis via the linear map lemma). Then,

$$\begin{aligned}
\|Sv\|^2 &= \| S(\langle v, e_1 \rangle e_1 + \cdots + \langle v, e_n \rangle e_n) \|^2 \\
&= \| \langle v, e_1 \rangle f_1 + \cdots + \langle v, e_n \rangle f_n \|^2 \\
&= |\langle v, e_1 \rangle|^2 + \cdots + |\langle v, e_n \rangle|^2 \\
&= \|v\|^2
\end{aligned}$$

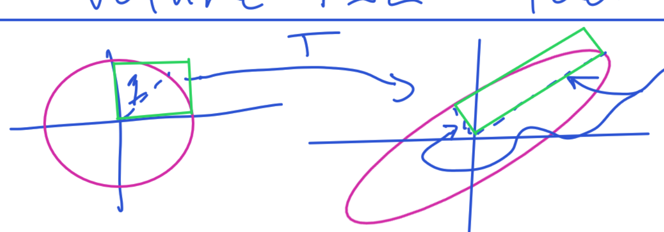
So, S is unitary! Moreover, $T = S\sqrt{T^*T}$.

Hence, $\det T = \det(S\sqrt{T^*T}) = (\det S)(\det \sqrt{T^*T}) = \det(\sqrt{T^*T}) = s_1 \cdots s_n$.

Theorem If $T \in \mathcal{L}(\mathbb{R}^n)$ & $\Omega \subseteq \mathbb{R}^n$. Then,

$$\text{volume } T\Omega = |\det T| \text{ volume } \Omega.$$

Picture



lengths are singular values

So, $|\det T|$ measures change in volume!