

8D Trace: A Connection Between Matrices and Operators

8.47 definition: trace of a matrix

Suppose A is a square matrix with entries in F . The *trace* of A , denoted by $\text{tr} A$, is defined to be the sum of the diagonal entries of A .

EX Let $A = \begin{bmatrix} \underline{3} & -1 & 2 \\ 3 & \underline{2} & -3 \\ 1 & 2 & \underline{0} \end{bmatrix}$. Then, $\text{tr} A = 3 + 2 + 0 = 5$.

Easy stuff!

8.49 trace of AB equals trace of BA

Suppose A is an m -by- n matrix and B is an n -by- m matrix. Then

$$\text{tr}(AB) = \text{tr}(BA).$$

Proof Idea: If $A = \begin{bmatrix} A_{11} & \dots & A_{1n} \\ \vdots & & \vdots \\ A_{m1} & \dots & A_{mn} \end{bmatrix}$ & $B = \begin{bmatrix} B_{11} & \dots & B_{1n} \\ \vdots & & \vdots \\ B_{n1} & \dots & B_{nn} \end{bmatrix}$
 Then $\text{tr} A = \text{tr} \tilde{A}$ & $\text{tr} B = \text{tr} \tilde{B}$ w/ $\tilde{A} = \begin{bmatrix} A_{11} & 0 & \dots & 0 \\ 0 & \ddots & & \\ 0 & \dots & 0 & A_{nn} \end{bmatrix}$ $\tilde{B} = \begin{bmatrix} B_{11} & 0 & \dots & 0 \\ 0 & \ddots & & \\ 0 & \dots & 0 & B_{nn} \end{bmatrix}$
 Since $\tilde{A}$ & $\tilde{B}$ commute, we're done!

more precisely: $(AB)_{ij} = \sum_{k=1}^n A_{ik} B_{kj}$, so

$$\text{tr} AB = \sum_{i=1}^m \sum_{k=1}^n A_{ik} B_{ki} = \sum_{k=1}^n \sum_{i=1}^m B_{ki} A_{ik} = \text{tr} BA.$$

8.50 trace of matrix of operator does not depend on basis

Suppose $T \in \mathcal{L}(V)$. Suppose $u_1, \dots, u_n$ and $v_1, \dots, v_n$ are bases of V . Then

$$\text{tr} M(T, (u_1, \dots, u_n)) = \text{tr} M(T, (v_1, \dots, v_n)). \quad \checkmark$$

Proof Let $A = M(T, (u_1, \dots, u_n))$ & $B = M(T, (v_1, \dots, v_n))$.

By the change of Basis formula (back in ch3), there is an invertible matrix C such that $A = C^{-1}BC$. Thus, using the previous result, $\text{tr}(A) = \text{tr}(C^{-1}BC) = \text{tr}(C^{-1}CB) = \text{tr}(B)$.

8.51 definition: trace of an operator

Suppose $T \in \mathcal{L}(V)$. The trace of T , denote $\text{tr } T$, is defined by

$$\text{tr } T = \text{tr } \mathcal{M}(T, (v_1, \dots, v_n)),$$

where $v_1, \dots, v_n$ is any basis of V .

8.52 on complex vector spaces, trace equals sum of eigenvalues

Suppose $\mathbf{F} = \mathbf{C}$ and $T \in \mathcal{L}(V)$. Then $\text{tr } T$ equals the sum of the eigenvalues of T , with each eigenvalue included as many times as its multiplicity.

Proof Using the Jordan form, there is a basis of V for which the matrix of T is upper triangular & has on its diagonal its eigenvalues repeated as many times as their multiplicities. So, since the trace does not depend on the choice of basis, using this basis delivers the result!

Jordan Form $M(T) = \begin{bmatrix} A_1 & 0 \\ & \ddots \\ 0 & A_m \end{bmatrix}$ w/ $A_i = \begin{bmatrix} \lambda_i & * \\ & \ddots \\ 0 & \lambda_i \end{bmatrix}$ $\dim G(\lambda_i, T)$

← This says that if $\lambda_1, \dots, \lambda_m$ are the distinct e-vals of T , then

$$\text{tr } T = \sum_{i=1}^m \lambda_i \cdot \dim(T - \lambda_i I)^{\dim V}$$

$$= \sum_{i=1}^m \lambda_i \cdot \dim G(\lambda_i, T)$$

8.54 trace and characteristic polynomial

Suppose $\mathbf{F} = \mathbf{C}$ and $T \in \mathcal{L}(V)$. Let $n = \dim V$. Then $\text{tr } T$ equals the negative of the coefficient of z^{n-1} in the characteristic polynomial of T .

Proof § $\mathbf{F} = \mathbf{C}, T \in \mathcal{L}(V), n = \dim V$, & $\lambda_1, \dots, \lambda_m$ are the eigenvalues of T . Then, the characteristic poly. of T is $(z - \lambda_1)^{\dim G(\lambda_1, T)} \dots (z - \lambda_m)^{\dim G(\lambda_m, T)}$.

Expanding, we get a poly of the form

not really showing this bit

$$z^n - \underbrace{(\underbrace{\lambda_1 + \dots + \lambda_1}_{\dim G(\lambda_1, T)} + \dots + \underbrace{\lambda_m + \dots + \lambda_m}_{\dim G(\lambda_m, T)})}_{\text{tr } T} z^{n-1} + \dots + ((\underbrace{\lambda_1 \dots \lambda_1}_{\dim G(\lambda_1, T)} \dots \underbrace{\lambda_m \dots \lambda_m}_{\dim G(\lambda_m, T)})) (-1)^n$$

or, if we let $\lambda_1, \dots, \lambda_n$ be the eigenvalues of T w/ repetition according to multiplicity, we get the slightly cleaner form:

$$z^n - (\lambda_1 + \dots + \lambda_n) z^{n-1} + \dots + (-1)^n (\lambda_1 \dots \lambda_n)$$

9C Determinants

Defining the Determinant

Defn $\S T \in \mathcal{L}(V)$. If $\mathbb{F} = \mathbb{C}$, then the determinant of T , denoted $\det T$, is the product of the eigenvalues of T , each repeated according to their multiplicity. If $\mathbb{F} = \mathbb{R}$, then $\det(T) = \det(T_{\mathbb{C}})$.

Theorem The determinant of $T \in \mathcal{L}(V)$ is, up to a sign, the constant term of the characteristic poly of T .

Proof (Look at the last one)

Theorem An operator is invertible if & only if its determinant is nonzero.

Proof By definition, $\det T = 0$ for $T \in \mathcal{L}(V)$ if & only if 0 is an eigenvalue of T (since a product of complex #'s is zero if & only if one is).
Hence, $\det T \neq 0 \Leftrightarrow 0$ not an e-val of T
 $\Leftrightarrow T$ is invertible.

Theorem $\S T \in \mathcal{L}(V)$. Then, the characteristic poly. of T is $\det(zI - T)$.

Proof $\S V$ is a $\mathbb{C}$ -V.S. $\S T \in \mathcal{L}(V)$, $\leftarrow$ If $\mathbb{F} = \mathbb{R}$, then the same argument holds w/ T replaced by $T_{\mathbb{C}}$
 λ is an e-val of $T \Leftrightarrow z - \lambda$ is an e-val of $zI - T$
(since $-(T - \lambda I) = (zI - T) - (z - \lambda)I$). So,
 $\det(zI - T) = (z - \lambda_1) \cdots (z - \lambda_n)$ where the λ_i 's are e-vals of T repeated according to their multiplicity. But this is just the characteristic poly of T !

Theorem Let $T \in \mathcal{L}(V)$ & define $g: \mathbb{F} \rightarrow \mathbb{F}$ by
 $g(x) = \det(I + xT)$.

Then, $g'(0) = \text{tr } T$.

Proof Notice that $I + xT$ has eigenvalues
 $1 + x\lambda_1, \dots, 1 + x\lambda_n$ (each repeated according to their multiplicity)

So, $g(x) = (1 + x\lambda_1) \cdots (1 + x\lambda_n)$ and

$$\begin{aligned} g'(0) &= \lim_{x \rightarrow 0} \frac{g(x) - g(0)}{x - 0} \\ &= \lim_{x \rightarrow 0} \frac{(1 + x\lambda_1) \cdots (1 + x\lambda_n) - 1}{x} \\ &= \lim_{x \rightarrow 0} \frac{\cancel{1} + (x\lambda_1 + \cdots + x\lambda_n) + \text{has } x \text{ terms of deg 2 or more} \cdots + (x\lambda_1) \cdots (x\lambda_n) - \cancel{1}}{x} \\ &= \lim_{x \rightarrow 0} (\lambda_1 + \cdots + \lambda_n) + (\text{stuff w/ } x\text{'s}) \\ &= \lambda_1 + \cdots + \lambda_n \\ &= \text{tr } T. \end{aligned}$$

So, the trace is the rate of change of the determinant as it starts moving from the identity in the direction of your operator

Fact: There is a fancy version of this known as Jacobi's formula, which says that if $A: \mathbb{R} \rightarrow \mathcal{L}(V)$, then $\frac{d}{dt} \det A(t) = \text{tr} \left(\text{adj}(A(t)) \frac{dA}{dt}(t) \right)$ where $\text{adj } A$ is the adjugate of A .