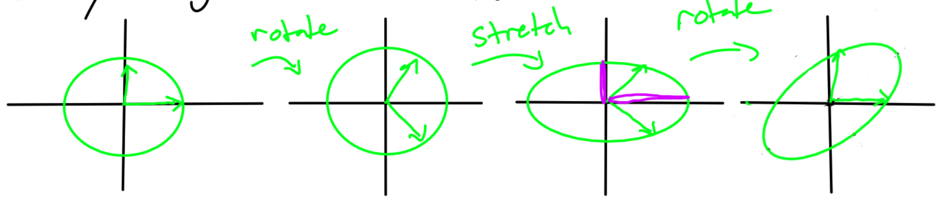
7E Singular Value Decomposition

Idea For linear maps between $\mathbb{R}^n$ & $\mathbb{R}^m$, everything works like this:



Today, we will show this is true & that an analog of it holds for maps in $\mathcal{L}(V,W)$ for V, W inner product spaces of finite dim.

7.64 *properties of T^*T*

Suppose $T \in \mathcal{L}(V,W)$. Then

(a) T^*T is a positive operator on V;

(b) $\operatorname{null} T^*T = \operatorname{null} T$;

(c) $\operatorname{range} T^*T = \operatorname{range} T^*$;

(d) $\dim \operatorname{range} T = \dim \operatorname{range} T^* = \dim \operatorname{range} T^*T$.

Proof **(a)** Notice $(T^*T)^* = T^*(T^*)^* = T^*T$, so T^*T is self-adjoint. If $v \in V$, $\langle T^*Tv, v\rangle = \langle Tv, Tv\rangle = \|Tv\|^2 \geq 0$. Hence, T^*T is positive.

(b) If $v \in \operatorname{null} T$, $Tv = 0$ so $T^*Tv = T^*0 = 0$ & $v \in \operatorname{null} T^*T$. Hence, $\underline{\operatorname{null} T \subseteq \operatorname{null} T^*T}$. Now, \$ $v \in \operatorname{null} T^*T$. Then, we have

$0 = \langle 0, v\rangle = \langle T^*Tv, v\rangle = \langle Tv, Tv\rangle = \|Tv\|^2$. So, $Tv = 0$ & $v \in \operatorname{null} T$. Hence $\underline{\operatorname{null} T^*T \subseteq \operatorname{null} T}$. So, $\operatorname{null} T = \operatorname{null} T^*T$.

(c) Since T^*T is self-adjoint (by part (a)), we have

$\operatorname{range} T^*T = (\operatorname{null} T^*T)^\perp = (\operatorname{null} T)^\perp = \operatorname{range} T^*$.

(d) By part (c), $\dim \operatorname{range} T^* = \dim \operatorname{range} T^*T$. Now, observe

$\dim \operatorname{range} T = \dim (\operatorname{null} T^*)^\perp = \dim W - \dim \operatorname{null} T^* = \dim \operatorname{range} T^*$.

7.65 definition: *singular values*

Suppose $T \in \mathcal{L}(V, W)$. The *singular values* of T are the nonnegative square roots of the eigenvalues of T^*T, listed in decreasing order, each included as many times as the dimension of the corresponding eigenspace of T^*T.

i.e. if $\lambda_1, \dots, \lambda_K$ are the eigenvalues of T^*T, and $\dim E(\lambda_i, T^*T) = n_i$, then $\underbrace{\sqrt{\lambda_1}, \dots, \sqrt{\lambda_1}}_{n_1}, \dots, \underbrace{\sqrt{\lambda_K}, \dots, \sqrt{\lambda_K}}_{n_K}$ are the singular values of T, assuming that $\lambda_1 \geq \dots \geq \lambda_K$.

7.68 *role of positive singular values*

Suppose that $T \in \mathcal{L}(V, W)$. Then

(a) T is injective $\iff$ 0 is not a singular value of T;

(b) the number of positive singular values of T equals $\dim \operatorname{range} T$;

(c) T is surjective $\iff$ number of positive singular values of T equals $\dim W$.

$\boxed{\text{Proof}}$ $\boxed{\text{(a)}}$ A linear map $T \in \mathcal{L}(V,W)$ is injective iff $\operatorname{null} T = \{0\}$.

So, T is injective $\iff \operatorname{null} T = \{0\} \iff \operatorname{null} T^*T = \{0\} \iff$ 0 not an eigenvalue of T^*T $\iff$ 0 not a singular value of T

$\boxed{\text{(b)}}$ Using the spectral theorem on T^*T, there is a basis of e-vectors of T^*T s.t. $M(T)$ is diagonal w/ its eigenvalues on the diagonal. Hence, $\dim \operatorname{range} T^*T = \dim \operatorname{span}(\text{columns of } M(T))$

$= \#$ of columns $- \#$ of times 0 appears on the diagonal of $M(T)$

$= \#$ of positive e-vals on diagonal of $M(T)$

$= \#$ of positive singular values

Now, just use $\dim \operatorname{range} T = \dim \operatorname{range} T^*T$

$\boxed{\text{(c)}}$ Obvious from (b).

list of eigenvalues	list of singular values
context: vector spaces	context: inner product spaces
defined only for linear maps from a vector space to itself (i.e. operators)	defined for linear maps from an inner product space to a possibly different inner product space
can be arbitrary real numbers (if $\mathbf{F} = \mathbf{R}$) or complex numbers (if $\mathbf{F} = \mathbf{C}$)	are nonnegative numbers
can be the empty list if $\mathbf{F} = \mathbf{R}$	length of list equals dimension of domain
includes 0 $\iff$ operator is not invertible	includes 0 $\iff$ linear map is not injective
no standard order, especially if $\mathbf{F} = \mathbf{C}$	always listed in decreasing order

bookkeeping, not really a "fact" in any substantial sense.

7.69 *isometries characterized by having all singular values equal* 1

Suppose that $S \in \mathcal{L}(V, W)$. Then

$$S \text{ is an isometry} \iff \text{all singular values of } S \text{ equal } 1.$$

$\boxed{\text{proof}}$ S is an isometry $\iff S^*S = I$

$\iff S^*S$ has all eigenvalues equal to 1

$\iff S$ has all singular values equal to 1

(the first equivalence is something we proved last time).

7.70 *singular value decomposition*

Suppose $T \in \mathcal{L}(V, W)$ and the positive singular values of T are $s_1, \ldots, s_m$. Then there exist orthonormal lists $e_1, \ldots, e_m$ in V and $f_1, \ldots, f_m$ in W such that

7.71 $$Tv = s_1 \langle v, e_1 \rangle f_1 + \cdots + s_m \langle v, e_m \rangle f_m$$

for every $v \in V$.

$\boxed{\text{proof}}$ Let $s_1, \ldots, s_n$ be the singular values of T (hence, $n = \dim V$). By the spectral theorem applied to the positive operator T^*T, there is an ONB $e_1, \ldots, e_n$ of V s.t. $T^*Te_k = s_k^2 e_k$ (since s_k is the square root of an eigenvalue of T^*T). Now, for $k \in \{1, \ldots, m\}$ (where $s_1, \ldots, s_m$ are positive & $s_{m+1}, \ldots, s_n$ are all 0), set $f_k = \frac{Te_k}{s_k}$.

Then, if $j, k \in \{1, \ldots, m\}$, we have

$$\langle f_j, f_k \rangle = \langle \tfrac{1}{s_j} Te_j, \tfrac{1}{s_k} Te_k \rangle = \frac{1}{s_j s_k} \langle e_j, T^*Te_k \rangle = \frac{s_k^2}{s_j s_k} \langle e_j, e_k \rangle = \begin{cases} 1 & j = k \\ 0 & j \neq k. \end{cases}$$

So, $f_1, \ldots, f_m$ is an orthonormal list in W.

Fix $v \in V$. Since $e_1, \ldots, e_n$ is an ONB, we get

$$v = \langle v, e_1 \rangle e_1 + \cdots + \langle v, e_n \rangle e_n$$

$$Tv = \langle v, e_1 \rangle Te_1 + \cdots + \langle v, e_n \rangle Te_n$$

$$Tv = s_1 \langle v, e_1 \rangle f_1 + \cdots + s_m \langle v, e_n \rangle f_m$$

since $s_k f_k = Te_k$ & this works if $s_k = 0$ as well since $Te_k = 0$ b/c $\operatorname{null} T = \operatorname{null} T^*T$

spectral theorem	**singular value decomposition**
describes only self-adjoint operators (when $\mathbf{F} = \mathbf{R}$) or normal operators (when $\mathbf{F} = \mathbf{C}$)	describes arbitrary linear maps from an inner product space to a possibly different inner product space
produces a single orthonormal basis	produces two orthonormal lists, one for domain space and one for range space, that are not necessarily the same even when range space equals domain space
different proofs depending on whether $\mathbf{F} = \mathbf{R}$ or $\mathbf{F} = \mathbf{C}$	same proof works regardless of whether $\mathbf{F} = \mathbf{R}$ or $\mathbf{F} = \mathbf{C}$

7.75 *singular value decomposition of adjoint and pseudoinverse*

Suppose $T \in \mathcal{L}(V, W)$ and the positive singular values of T are $s_1, \ldots, s_m$. Suppose $e_1, \ldots, e_m$ and $f_1, \ldots, f_m$ are orthonormal lists in V and W such that

7.76 $$Tv = s_1 \langle v, e_1 \rangle f_1 + \cdots + s_m \langle v, e_m \rangle f_m$$

for every $v \in V$. Then

7.77 $$T^* w = \underbrace{s_1 \langle w, f_1 \rangle e_1}_{\text{this}/s_1^2} + \cdots + \underbrace{s_m \langle w, f_m \rangle e_m}_{\text{this}/s_m^2}$$

and

7.78 $$T^\dagger w = \frac{\langle w, f_1 \rangle}{s_1} e_1 + \cdots + \frac{\langle w, f_m \rangle}{s_m} e_m$$

for every $w \in W$.

7.80 *matrix version of SVD*

Suppose A is an M-by-n matrix of rank $m \geq 1$. Then there exist an M-by-m matrix B with orthonormal columns, an m-by-m diagonal matrix D with positive numbers on the diagonal, and an n-by-m matrix C with orthonormal columns such that

$$A = BDC^*.$$

This is the picture from the start of class!