Isometries

7.44 definition: *isometry*

A linear map $S \in \mathcal{L}(V, W)$ is called an *isometry* if

$$\|Sv\|_W = \|v\|_V$$

for every $v \in V$. In other words, a linear map is an isometry if it preserves norms.

$\boxed{\text{Note}}$ Suppose $Sv = 0$. Then, $\|v\| = \|Sv\| = 0$, so $v = 0$. That is, $\operatorname{null} S = \{0\}$ & S is injective!

7.49 *characterization of isometries*

Suppose $S \in \mathcal{L}(V, W)$. Suppose $e_1, \dots, e_n$ is an orthonormal basis of V and $f_1, \dots, f_m$ is an orthonormal basis of W. Then the following are equivalent.

(a) S is an isometry.

(b) $S^*S = I$.

(c) $\langle Su, Sv \rangle = \langle u, v \rangle$ for all $u, v \in V$.

(d) $Se_1, \dots, Se_n$ is an orthonormal list in W.

(e) The columns of $\mathcal{M}(S, (e_1, \dots, e_n), (f_1, \dots, f_m))$ form an orthonormal list in $\mathbf{F}^m$ with respect to the Euclidean inner product.

$\boxed{\text{Proof}}$ $\boxed{(a) \Rightarrow (b)}$ If $v \in V$, then

$$\begin{aligned}\langle (I - S^*S)v, v \rangle &= \langle v, v \rangle - \langle S^*Sv, v \rangle \\ &= \langle v, v \rangle - \langle Sv, Sv \rangle \\ &= \|v\|^2 - \|Sv\|^2 \\ &= 0\end{aligned}$$

Since $I - S^*S$ is self-adjoint & satisfies $\langle (I - S^*S)v, v \rangle = 0$ for $\forall v \in V$.

So, $I - S^*S$ is the zero operator, that is $S^*S = I$.

$\boxed{(b) \Rightarrow (c)}$ Suppose $S^*S = I$. $\langle Su, Sv \rangle = \langle S^*Su, v \rangle = \langle u, v \rangle$

$\boxed{(c) \Rightarrow (d)}$ If $e_1, \dots, e_n$ is an ONB of V & $\langle Su, Sv \rangle = \langle u, v \rangle$ $\forall u, v \in V$, then $\langle Se_i, Se_j \rangle = \langle e_i, e_j \rangle = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$. So, $Se_1, \dots, Se_n$ is an orthonormal list in W.

$\boxed{(d) \Rightarrow (e)}$ Let $A = \mathcal{M}(S)$. Then, for $k, r \in \{1, \dots, n\}$, we have

$$A_{\cdot,k} \bullet A_{\cdot,r} = \sum_{j=1}^{n} A_{j,k} \overline{A_{j,r}} = \left\langle \sum_{j=1}^{m} A_{jk} f_j, \sum_{i=1}^{m} A_{ir} f_i \right\rangle = \langle Se_k, Se_r \rangle = \begin{cases} 1 & k = r \\ 0 & k \neq r \end{cases}$$

$$\left\langle \sum_{j=1}^{m} A_{jk} f_j, \sum_{i=1}^{m} A_{ir} f_i \right\rangle = \sum_{j=1}^{m} A_{jk} \left\langle f_j, \sum_{i=1}^{m} A_{ir} f_i \right\rangle = \sum_{j=1}^{m} A_{jk} \sum_{i=1}^{m} \overline{A_{ir}} \langle f_j, f_i \rangle$$

$\boxed{\text{Proof}}$ (continued)

$\boxed{(e) \Rightarrow (a)}$ If the columns of $M(S)$ are an orthonormal list in $\mathbb{F}^m$, then $Se_1, \ldots, Se_n$ is an orthonormal list in W.

So, for $v \in V$, $v = \langle v, e_1 \rangle e_1 + \cdots + \langle v, e_n \rangle e_n$ &

$Sv = \langle v, e_1 \rangle Se_1 + \cdots + \langle v, e_n \rangle Se_n$. So, $\|v\|^2 = |\langle v, e_1 \rangle|^2 + \cdots + |\langle v, e_n \rangle|^2$

& $\|Sv\|^2 = |\langle v, e_1 \rangle|^2 + \cdots + |\langle v, e_n \rangle|^2$ & S is an isometry.

Unitary Operators

7.51 definition: *unitary operator*

An operator $S \in \mathcal{L}(V)$ is called *unitary* if S is an invertible isometry.

7.53 *characterization of unitary operators*

Suppose $S \in \mathcal{L}(V)$. Suppose $e_1, \ldots, e_n$ is an orthonormal basis of V. Then the following are equivalent.

(a) S is a unitary operator.

(b) $S^*S = SS^* = I$.

(c) S is invertible and $S^{-1} = S^*$.

(d) $Se_1, \ldots, Se_n$ is an orthonormal basis of V.

(e) The rows of $\mathcal{M}(S, (e_1, \ldots, e_n))$ form an orthonormal basis of $\mathbf{F}^n$ with respect to the Euclidean inner product.

(f) S^* is a unitary operator.

$\boxed{\text{Proof}}$ $\boxed{(a) \Rightarrow (b)}$ If S is unitary, it is an isometry. So, $S^*S = I$. Since unitary operators are invertible, this gives $S^* = S^*SS^{-1} = S^{-1}$. So, $S^*S = SS^* = I$.

$\boxed{(b) \Rightarrow (c)}$ & $S^*S = SS^* = I$. So, by definition, $S^{-1} = S^*$

$\boxed{(c) \Rightarrow (d)}$ Since $S^{-1} = S^*$ implies $S^*S = I$, the previous result delivers this via (b)$\Leftrightarrow$(d) in the last result.

$\boxed{(d) \Rightarrow (e)}$ This is a direct consequence of the last result (particularly, (d)$\Rightarrow$(e) there).

$\boxed{(e) \Rightarrow (f)}$ $M(S)$ has ~~cols~~ rows an ONB implies $M(S^*) = \overline{M(S)}^t$ cols an ONB. So, S^* is an isometry & given both the rows & columns of $M(S)$ are ONBs, S^* is invertible. (doing this again w/ S^* replacing S)

$\boxed{(f) \Rightarrow (a)}$

just do (a)$\Rightarrow$(f) to S^* to get $(S^*)^* = S$ is unitary

7.54 *eigenvalues of unitary operators have absolute value* 1

Suppose λ is an eigenvalue of a unitary operator. Then $|\lambda| = 1$.

$\boxed{\text{Proof}}$ Suppose S is unitary $\ni$ $Sv = \lambda v$ for $v \neq 0$ & $\lambda \in \mathbb{F}$. Then,

$$1 = \frac{\|Sv\|}{\|v\|} = \frac{\|\lambda v\|}{\|v\|} = \frac{|\lambda|\,\|v\|}{\|v\|} = |\lambda|, \text{ since } \|Sv\| = \|v\|.$$

7.55 *description of unitary operators on complex inner product spaces*

Suppose $\mathbf{F} = \mathbf{C}$ and $S \in \mathcal{L}(V)$. Then the following are equivalent.

(a) S is a unitary operator.

(b) There is an orthonormal basis of V consisting of eigenvectors of S whose corresponding eigenvalues all have absolute value 1.

$\boxed{\text{Proof}}$ Suppose S is unitary. Then, $S^*S = I = SS^*$, so S is normal. Thus, by the complex spectral theorem $\exists$ an ONB of e-vec of S. Since all e-values of S have modulus 1, (a)$\Rightarrow$(b).

Now, suppose $\exists$ an ONB of e-vectors of S w/ corresponding e-vals all modulus 1. Then, $\lambda_1 e_1, \dots, \lambda_n e_n$ is also an ONB, since $\langle \lambda_i e_i, \lambda_k e_k \rangle = \langle e_i, e_k \rangle \lambda_i \overline{\lambda_k} = \begin{cases} 0 & i \neq k \\ \underset{\underset{1}{\shortparallel}}{|\lambda_i|^2} & i = k \end{cases}$

So, since $\lambda_1 e_1, \dots, \lambda_n e_n$ is just $Se_1, \dots, Se_n$, S is unitary by the previous result.

QR Factorization

7.56 definition: *unitary matrix*

An n-by-n matrix is called *unitary* if its columns form an orthonormal list in $\mathbf{F}^n$.

← i.e. if it is the matrix of a unitary operator w.r.t. an ONB.

7.57 *characterizations of unitary matrices*

Suppose Q is an n-by-n matrix. Then the following are equivalent.

(a) Q is a unitary matrix.

(b) The rows of Q form an orthonormal list in $\mathbf{F}^n$.

(c) $\|Qv\| = \|v\|$ for every $v \in \mathbf{F}^n$.

(d) $Q^*Q = QQ^* = I$, the n-by-n matrix with 1's on the diagonal and 0's elsewhere.

7.58 *QR factorization*

Suppose A is a square matrix with linearly independent columns. Then there exist ~~unique~~ matrices Q and R such that Q is unitary, R is upper triangular with only positive numbers on its diagonal, and

$$A = QR.$$

True, but we won't prove it

proof Let $A \in \mathbb{F}^{n\times n}$ have linearly indep. cols, $v_1, \dots, v_n$. Use GS to get $e_1, \dots, e_n$ an ONB of $\mathbb{F}^n$ s.t. $\operatorname{Span}(v_1, \dots, v_k) = \operatorname{Span}(e_1, \dots, e_k)$. Let R be the matrix w/ $R_{jk} = \langle v_k, e_j \rangle$. Then, R is UT, since for $k<j$, e_j is orthogonal to everything in $\operatorname{Span}(v_1, \dots, v_k) = \operatorname{Span}(e_1, \dots, e_k)$. Then, if Q is the matrix w/ columns $e_1, \dots, e_n$, it is unitary (since its cols are an ONB) & the kth col of QR is just $\langle v_k, e_1 \rangle e_1 + \dots + \langle v_k, e_k \rangle e_k = v_k$. So, $QR = A$.

$\boxed{\text{Ex}}$ Let $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -4 \\ 0 & 3 & 2 \end{bmatrix}$. Applying GS to

$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $v_2 = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$, $v_3 = \begin{bmatrix} 1 \\ -4 \\ 2 \end{bmatrix}$, we get

$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $e_2 = \begin{bmatrix} 1 \\ 1/\sqrt{10} \\ -3/\sqrt{10} \end{bmatrix}$, $e_3 = \begin{bmatrix} 0 \\ 3/\sqrt{10} \\ 1/\sqrt{10} \end{bmatrix}$. Then,

$$Q = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1/\sqrt{10} & 3/\sqrt{10} \\ 0 & -3/\sqrt{10} & 1/\sqrt{10} \end{bmatrix} \; \& \; R = \begin{bmatrix} \langle v_1, e_1 \rangle & \langle v_2, e_1 \rangle & \langle v_3, e_1 \rangle \\ \langle v_1, e_2 \rangle & \langle v_2, e_2 \rangle & \langle v_3, e_2 \rangle \\ \langle v_1, e_3 \rangle & \langle v_2, e_3 \rangle & \langle v_3, e_3 \rangle \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & \sqrt{10} & \frac{\sqrt{10}}{5} \\ 0 & 0 & \frac{7\sqrt{10}}{5} \end{bmatrix}$$

You can check that $A = QR$.

This is nice: If we want to solve $Ax = b$, then $QRx = b$ & $Rx = Q^* b$ since Q is unitary.

As $Rx = y$ is easy to solve (R is UT), we can do this quickly!

$$Rx = \begin{bmatrix} 1 & 2 & 1 \\ 0 & \sqrt{10} & \frac{\sqrt{10}}{5} \\ 0 & 0 & \frac{7\sqrt{10}}{5} \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a + 2b + c \\ \sqrt{10}\, b + \frac{\sqrt{10}}{5} c \\ \frac{7\sqrt{10}}{5} c \end{bmatrix}$$

So, we can solve $Rx = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$ via $c = \frac{5}{7\sqrt{10}} z_3$

$$b = \frac{\left(z_2 - \left(\frac{\sqrt{10}}{5}\right)\left(\frac{5}{7\sqrt{10}}\right) z_3\right)}{\sqrt{10}}$$

$a = $ (same process)