7C Positive Operators

7.34 definition: *positive operator*

An operator $T \in \mathcal{L}(V)$ is called *positive* if T is self-adjoint and

$$\boxed{\langle Tv, v \rangle} \geq 0$$

$\mathbb{R}$

for all $v \in V$.

Note: If V is an inner product space over $\mathbb{C}$, then we can drop the requirement that T is self-adjoint since $\langle Tv, v \rangle \in \mathbb{R}$ for all $v \in V$ implies this in that case. Be careful to not make a mistake and forget this! $\mathbb{C}$ doesn't have "$\leq$" like $\mathbb{R}$ does!

7.36 definition: *square root*

An operator R is called a *square root* of an operator T if $R^2 = T$.

Ex: Let $T \in \mathcal{L}(\mathbb{F}^3)$ be $T(x_1, x_2, x_3) = (x_3, 0, 0)$.
Let $R \in \mathcal{L}(\mathbb{F}^3)$ be $R(x_1, x_2, x_3) = (x_2, x_3, 0)$. Then,
$R^2(x_1, x_2, x_3) = R(x_2, x_3, 0) = (x_3, 0, 0) = T(x_1, x_2, x_3)$.
So, $R^2 = T$ & R is a square root of T.

Notice: If $R'(x_1, x_2, x_3) = (-x_2, -x_3, 0)$, then
$(R')^2(x_1, x_2, x_3) = R'(-x_2, -x_3, 0) = (x_3, 0, 0) = T(x_1, x_2, x_3)$,
so R' is a square root of T as well.

7.38 *characterization of positive operators*

Let $T \in \mathcal{L}(V)$. Then the following are equivalent.

(a) T is a positive operator.

(b) T is self-adjoint and all eigenvalues of T are nonnegative.

(c) With respect to some orthonormal basis of V, the matrix of T is a diagonal matrix with only nonnegative numbers on the diagonal.

(d) T has a positive square root.

(e) T has a self-adjoint square root.

(f) $T = R^*R$ for some $R \in \mathcal{L}(V)$.

Proof ((a) $\Rightarrow$ (b)) That a positive operator is self adjoint is immediate from the definition. Suppose T is positive & λ is an e-val of T w/ e-vec v. Then $0 \le \langle Tv, v\rangle = \langle \lambda v, v\rangle = \lambda \|v\|^2$

So, $\lambda \ge 0$.

((b) $\Rightarrow$ (c)) Spectral Theorem

((c) $\Rightarrow$ (d)) By the linear map lemma, there is a $R \in \mathcal{L}(V)$ s.t. $Re_k = \sqrt{\lambda_k}\, e_k$ where $e_1, \dots, e_n$ is the assumed ONB of V s.t. $\mathcal{M}(T)$ is diagonal w/ λ_k its kth diagonal entry. Certainly, R is positive: $\langle Re_k, e_k\rangle = \sqrt{\lambda_k}\langle e_k, e_k\rangle = \sqrt{\lambda_k}\,\|e_k\|^2 = \sqrt{\lambda_k} \ge 0$.

Moreover, $R^2 e_k = \sqrt{\lambda_k}^{\,2} e_k = \lambda_k e_k = Te_k$, so $T = R^2$.

((d) $\Rightarrow$ (e)) Positive operators are self adjoint.

((e) $\Rightarrow$ (f)) Suppose $T = R^2$ & R is self-adjoint. Then, $R = R^*$, so $R^2 = R^*R = T$.

((f) $\Rightarrow$ (a)) Suppose $T = R^*R$. Then, $T^* = R^*(R^*)^* = R^*R = T$.

So, T is self-adjoint. Moreover, $\langle Tv, v\rangle = \langle R^*Rv, v\rangle$

$= \langle Rv, Rv\rangle$

$= \|Rv\|^2 \ge 0$.

So, T is positive.

7.39 *each positive operator has only one positive square root*

Every positive operator on V has a unique positive square root.

$\boxed{\text{Proof}}$ Suppose $T \in \mathcal{L}(V)$ is a positive operator. Let R be a positive square root of T, which we know exists by the last result. We show uniqueness. Let λ be an eigenvalue of T w/ eigenvector v. We will show that $Rv = \sqrt{\lambda}\, v$ which shows R is unique, since V has a basis of eigenvectors of T (by the spectral theorem).

That $Rv = \sqrt{\lambda} v$ is true follows from applying the spectral theorem to R to get an ONB of e-vecs of R for V. Let $e_1, \dots, e_n$ be this basis. Since R is positive, $\exists$ nonnegative $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ s.t. $Re_k = \sqrt{\lambda_k}\, e_k$. So, write $v = a_1 e_1 + \dots + a_n e_n$, and notice $R^2 v = a_1 \lambda_1 e_1 + \dots + a_n \lambda_n e_n$.

& $Tv = \lambda v$ implies $\lambda v = a_1 \lambda e_1 + \dots + a_n \lambda e_n = Tv = R^2 v = a_1 \lambda_1 e_1 + \dots + a_n \lambda_n e_n$.

So, $a_1(\lambda - \lambda_1)e_1 + \dots + a_n(\lambda - \lambda_n)e_n = 0$ & since $e_1, \dots, e_n$ is a basis, if $a_i \neq 0$, then $\lambda = \lambda_i$. So, $Rv = a_1\sqrt{\lambda}e_1 + \dots + a_n\sqrt{\lambda}e_n = \sqrt{\lambda} v$

(since, if $a_i = 0$, we can still replace $\sqrt{\lambda_k}$ w/ $\sqrt{\lambda}$ here w/o changing anything).

7.40 notation: $\sqrt{T}$

For T a positive operator, $\sqrt{T}$ denotes the unique positive square root of T.

7.43 *T positive and $\langle Tv, v\rangle = 0 \implies Tv = 0$*

Suppose T is a positive operator on V and $v \in V$ is such that $\langle Tv, v\rangle = 0$. Then $Tv = 0$.

Proof $0 = \langle Tv, v\rangle = \langle \sqrt{T}\sqrt{T}v, v\rangle = \langle \sqrt{T}v, \sqrt{T}v\rangle = \|\sqrt{T}v\|$

So, $\sqrt{T}v = 0$. Thus, $Tv = \sqrt{T}\sqrt{T}v = \sqrt{T}0 = 0$.

Ex Let $T \in \mathcal{L}(\mathbb{R}^2)$ be $T(x,y) = (x+y, x+y)$.

$\langle T(x,y), (x,y)\rangle = \langle (x+y, x+y), (x,y)\rangle = x(x+y) + y(x+y) = (x+y)^2 \geq 0$

So, T is positive, since w.r.t. the standard basis $(1,0), (0,1)$ of $\mathbb{R}^2$, $\mathcal{M}(T) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \mathcal{M}(T)^t = \mathcal{M}(T^*)$. Now, $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}), (\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}})$ is a basis of $\mathbb{R}^2$ that consists only of e-vectors of T w/ corresponding e-vals $2, 0$. W.r.t. this basis, T has a matrix $\mathcal{M}(T) = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$, so $\mathcal{M}(\sqrt{T}) = \begin{bmatrix} \sqrt{2} & 0 \\ 0 & 0 \end{bmatrix}$ w.r.t. this basis.

W.r.t. the standard basis of $\mathbb{R}^2$, $\mathcal{M}(\sqrt{T}) = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$.