

7B Spectral Theorem

Real Spectral Theorem

7.26 invertible quadratic expressions

Suppose $T \in \mathcal{L}(V)$ is self-adjoint and $b, c \in \mathbf{R}$ are such that $b^2 < 4c$. Then

$$\underline{T^2 + bT + cI}$$

is an invertible operator.

proof $\S v \neq 0$. Then,

$$\begin{aligned} \langle (T^2 + bT + cI)v, v \rangle &= \langle T^2v, v \rangle + b \langle Tv, v \rangle + c \langle v, v \rangle \\ &= \langle Tv, T_v^* \rangle + b \langle Tv, v \rangle + c \langle v, v \rangle \\ &= \langle Tv, Tv \rangle + b \langle Tv, v \rangle + c \langle v, v \rangle \\ &= \|Tv\|^2 + b \langle Tv, v \rangle + c \|v\|^2 \\ &\geq \|Tv\|^2 - |b| | \langle Tv, v \rangle | + c \|v\|^2 \\ &\geq \|Tv\|^2 - |b| \|Tv\| \|v\| + c \|v\|^2 \\ &= \left(\|Tv\| - \frac{|b| \|v\|}{2} \right)^2 + \left(c - \frac{b^2}{4} \right) \|v\|^2 \\ &\geq \left(c - \frac{b^2}{4} \right) \|v\|^2 \end{aligned}$$

via CS.
- $| \langle v, v \rangle |^2 = \|v\|^2 \|v\|^2$

Since $b^2 < 4c$, i.e.

$$\begin{aligned} &4c - b^2 > 0 \\ &0 < c - \frac{b^2}{4} \end{aligned}$$

> 0

So, $(T^2 + bT + cI)v \neq 0$ if $v \neq 0$.

Hence, it is injective & thus invertible.

7.27 minimal polynomial of self-adjoint operator

Suppose $T \in \mathcal{L}(V)$ is self-adjoint. Then the minimal polynomial of T equals $(z - \lambda_1) \cdots (z - \lambda_m)$ for some $\lambda_1, \dots, \lambda_m \in \mathbb{R}$. ✓

Proof

First, if $F = \mathbb{C}$, then the zeros of the min. poly. of T are the eigenvalues of T , all of which are real (as we proved last time). So, since polynomials factor completely over $\mathbb{C}$, the min. poly. of T is of the desired form.

Now, if $F = \mathbb{R}$, the min. poly. of T is of the form $(z - \lambda_1) \cdots (z - \lambda_n) (z^2 + b_1 z + c_1) \cdots (z^2 + b_n z + c_n)$ for some $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ & $b_1, c_1, \dots, b_n, c_n \in \mathbb{R}$ s.t. $b_k^2 < 4c_k$ (since the roots of $z^2 + b_k z + c_k$ are $z = \frac{-b_k \pm \sqrt{b_k^2 - 4c_k}}{2}$, which is real unless $b_k^2 - 4c_k < 0$, i.e. $b_k^2 < 4c_k$). But this says

$$(T - \lambda_1 I) \cdots (T - \lambda_n I) (T^2 + b_1 T + c_1 I) \cdots (T^2 + b_n T + c_n I) = 0$$

and since $T^2 + b_n T + c_n I$ is invertible, we get (if there are any such quadratic terms) that $(T - \lambda_1 I) \cdots (T - \lambda_n I) (T^2 + b_1 T + c_1 I) \cdots (T^2 + b_{n-1} T + c_{n-1} I) = 0$. This would contradict the minimality of the degree of the minimal polynomial. Hence, there must be no quadratic terms & the min. poly. must be of the form $(z - \lambda_1) \cdots (z - \lambda_m)$ for some $\lambda_1, \dots, \lambda_m \in \mathbb{R}$.

7.29 real spectral theorem

Suppose $\mathbf{F} = \mathbf{R}$ and $T \in \mathcal{L}(V)$. Then the following are equivalent.

- (a) T is self-adjoint.
- (b) T has a diagonal matrix with respect to some orthonormal basis of V .
- (c) V has an orthonormal basis consisting of eigenvectors of T .

So, self-adjoint
is the same as
diagonalizable by
an orthonormal basis
of eigenvectors

Proof $\S$ (a) is true. Then, T has an UT matrix w.r.t. an orthonormal basis of V since its min. poly. is of the form $(z - \lambda_1) \cdots (z - \lambda_n)$ for some $\lambda_1, \dots, \lambda_n \in \mathbf{R}$ (this is an old result we proved in Ch 6). W.r.t. this basis, $M(T)$ is U.T. $\S$, since T is self adjoint, we get $M(T) = M(T^*) = M(T)^t$. However, $M(T)^t$ is lower triangular, while $M(T)$ is UT, so $M(T)$ must be diagonal. Therefore, $(a) \Rightarrow (b)$.

Now, $\S$ (b). Then, $M(T)^t = M(T^*) = M(T)$ (since $M(T) = M(T)^t$). Hence, $T = T^*$ $\S$ (b) $\Rightarrow$ (a).

That (c) $\Leftrightarrow$ (b) was shown in Chapter 5

(i.e. diagonalizability $\Leftrightarrow \exists$ of a basis of e-vectors of T for V)

Complex Spectral Theorem

7.31 complex spectral theorem

Suppose $\mathbf{F} = \mathbf{C}$ and $T \in \mathcal{L}(V)$. Then the following are equivalent.

- (a) T is normal.
- (b) T has a diagonal matrix with respect to some orthonormal basis of V .
- (c) V has an orthonormal basis consisting of eigenvectors of T .

Proof $\S$ (a). By Schur's theorem (from Ch 6), there is an orthonormal basis of V , $e_1, \dots, e_n$, s.t., $M(T)$ is upper triangular wrt. this basis. Let $M(T) = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ & \ddots & \vdots \\ 0 & & a_{nn} \end{bmatrix}$.

Then, $Te_1 = a_{11}e_1$, so $\|Te_1\|^2 = |a_{11}|^2$ $\S$ since $M(T^*) = M(T)^*$ (where the second $*$ denotes the conjugate transpose) $T^*e_1 = \overline{a_{11}}e_1 + \dots + \overline{a_{1n}}e_n$.

So, since $\|T\| = \|T^*\|$ when T is normal, we get

$$|a_{11}|^2 = \|Te_1\|^2 = \|T^*e_1\|^2 = |a_{11}|^2 + \dots + |a_{1n}|^2$$

Hence, $a_{12}, \dots, a_{1n}$ are all 0. Repeating this process for $e_2, \dots, e_n$ yields that the off-diagonal entries of $M(T)$ are all 0. Hence, it is diagonal $\S$ (a) $\Rightarrow$ (b).

Now, $\S$ (b). Then, T and T^* both have diagonal matrices wrt some orthonormal basis of V (in fact, wrt the same orthonormal basis of V). Since any two diagonal matrices commute, $M(TT^*) = M(T)M(T^*) = M(T^*)M(T) = M(T^*T)$. Therefore, $TT^* = T^*T$ $\S$ T commutes w/ its adjoint (that is, T is normal). So, (b) $\Rightarrow$ (a).

(c) $\Leftrightarrow$ (b) is the same as in the real case.