7.14 *$\langle Tv, v\rangle$ is real for all v $\iff$ T is self-adjoint (assuming $\mathbf{F} = \mathbf{C}$)*

Suppose V is a complex inner product space and $T \in \mathcal{L}(V)$. Then

$$T \text{ is self-adjoint} \iff \langle Tv, v\rangle \in \mathbf{R} \text{ for every } v \in V.$$

$\boxed{\text{proof}}$ If $v \in V$, then $\langle T^* v, v\rangle = \langle v, Tv\rangle = \overline{\langle Tv, v\rangle}$. As such, we get

$$\begin{aligned}
T \text{ is self-adjoint} &\iff T = T^* \\
&\iff T - T^* = 0 \\
&\iff \langle (T-T^*)v, v\rangle = 0 \text{ for all } v \in V \\
&\iff \langle Tv, v\rangle - \overline{\langle Tv, v\rangle} = 0 \text{ for all } v \in V \\
&\iff \langle Tv, v\rangle \in \mathbb{R} \text{ for all } v \in V
\end{aligned}$$

If $z = x+iy$, $\bar{z} = x - iy$
$\therefore z - \bar{z} = x + iy - x + iy$
$= 2iy$

7.16 *T self-adjoint and $\langle Tv, v\rangle = 0$ for all v $\iff$ $T = 0$*

Suppose T is a self-adjoint operator on V. Then

$$\langle Tv, v\rangle = 0 \text{ for every } v \in V \iff T = 0.$$

$\boxed{\text{proof}}$ We have already proved this in the case that $\mathbb{F} = \mathbb{C}$. So, assume V is a real inner product space. Then, if $u, w \in V$, we have

$$\langle Tu, w\rangle = \frac{\langle T(u+w), (u+w)\rangle - \langle T(u-w), (u-w)\rangle}{4}$$

So $\langle Tu, w\rangle$ is of the form of a sum of inner products of the form $\langle Tv, v\rangle$ for several v's

(which follows from $\langle Tw, u\rangle \underset{\text{self adjoint}}{=} \langle w, Tu\rangle \underset{\text{since } V \text{ is a real inner product space}}{=} \langle Tu, w\rangle$)

So, suppose $\langle Tv, v\rangle = 0$ for all $v \in V$. Then, $\langle Tu, w\rangle = 0$ for all $u, w \in V$ and this implies that $T = 0$.

The other direction is easy: If $T = 0$, then $\langle Tv, v\rangle = \langle 0, v\rangle = 0$, $\forall v \in V$.

Normal Operators

7.18 definition: *normal*

- An operator on an inner product space is called *normal* if it commutes with its adjoint.
- In other words, $T \in \mathcal{L}(V)$ is normal if $TT^* = T^*T$.

$\boxed{\text{Idea}}$ Every self-adjoint operator satisfies $T = T^*$, so certainly $T^*T = T^2 = TT^*$ (i.e. every self-adjoint operator is normal). So, in some sense, normality is just a way of weakening self-adjointness.

7.20 *T is normal if and only if Tv and T^*v have the same norm*

Suppose $T \in \mathcal{L}(V)$. Then

$$T \text{ is normal} \iff \|Tv\| = \|T^*v\| \text{ for every } v \in V.$$

$\boxed{\text{Proof}}$

$$\begin{aligned}
T \text{ normal} &\iff TT^* = T^*T \\
&\iff T^*T - TT^* = 0 \\
&\iff \langle (T^*T - TT^*)v, v\rangle = 0 \text{ for all } v \in V \\
&\iff \langle T^*Tv, v\rangle = \langle TT^*v, v\rangle \text{ for all } v \in V \\
&\iff \langle Tv, Tv\rangle = \langle T^*v, T^*v\rangle \text{ for all } v \in V \\
&\iff \|Tv\|^2 = \|T^*v\|^2 \text{ for all } v \in V \\
&\iff \|Tv\| = \|T^*v\| \text{ for all } v \in V
\end{aligned}$$

Notice that

$(T^*T - TT^*)^*$

$= T^*(T^*)^* - (T^*)^*T^*$

$= T^*T - TT^*$

Same! So, $T^*T - TT^*$ is self-adjoint!

7.21 *range, null space, and eigenvectors of a normal operator*

Suppose $T \in \mathcal{L}(V)$ is normal. Then

(a) $\operatorname{null} T = \operatorname{null} T^*$;

(b) $\operatorname{range} T = \operatorname{range} T^*$;

(c) $V = \operatorname{null} T \oplus \operatorname{range} T$;

(d) $T - \lambda I$ is normal for every $\lambda \in \mathbf{F}$;

(e) if $v \in V$ and $\lambda \in \mathbf{F}$, then $Tv = \lambda v$ if and only if $T^* v = \bar{\lambda} v$.

$\boxed{\text{Proof}}$ (a) using the last result, T is normal iff $\|Tv\| = \|T^*v\|$ for all $v \in V$. So, $v \in \operatorname{null} T \iff Tv = 0 \iff \|Tv\| = 0 \iff \|T^*v\| = 0 \iff T^*v = 0 \iff v \in \operatorname{null} T^*$

(b) $\operatorname{range} T = (\operatorname{null} T^*)^\perp = (\operatorname{null} T)^\perp = \operatorname{range} T^*$

(c) $V = \operatorname{null} T \oplus (\operatorname{null} T)^\perp = \operatorname{null} T \oplus \operatorname{range} T^* = \operatorname{null} T \oplus \operatorname{range} T$

(d) \$ $\lambda \in \mathbb{F}$. Then,
$$\begin{aligned}(T-\lambda I)(T-\lambda I)^* &= (T-\lambda I)(T^* - \bar{\lambda} I) \\ &= TT^* - \bar{\lambda} I T - \lambda I T^* + \lambda \bar{\lambda} I^2 \\ &= T^*T - \bar{\lambda} T - \lambda T^* + \lambda \bar{\lambda} I \\ &= (T^* - \bar{\lambda} I)(T - \lambda I) \\ &= (T-\lambda I)^*(T - \lambda I)\end{aligned}$$

(e) \$ $v \in V$ \$ $\lambda \in \mathbb{F}$. Then, $\|(T-\lambda I)v\| = \|(T-\lambda I)^* v\| = \|(T^* - \bar{\lambda} I)v\|$.

So, $(T-\lambda I)v = 0$ if \$ only if $(T^* - \bar{\lambda} I)v = 0$. That is, $Tv = \lambda v$ if \$ only if $T^* v = \bar{\lambda} v$.

7.22 *orthogonal eigenvectors for normal operators*

Suppose $T \in \mathcal{L}(V)$ is normal. Then eigenvectors of T corresponding to distinct eigenvalues are orthogonal.

Proof Suppose $\alpha, \beta \in \mathbb{F}$ are distinct e-vals of T & $u, w \in V$ are corresponding e-vecs (so that $Tu = \alpha u$, $Tw = \beta w$). Then, $T^*w = \bar{\beta} w$ (since $Tv = \lambda v \Rightarrow T^*v = \bar{\lambda} v$). Thus, we have

$$\begin{aligned}(\alpha - \beta)\langle u, w\rangle &= \alpha\langle u, w\rangle - \beta\langle u, w\rangle \\ &= \langle \alpha u, w\rangle - \langle \beta u, w\rangle \\ &= \langle \alpha u, w\rangle - \langle u, \bar{\beta} w\rangle \\ &= \langle Tu, w\rangle - \langle u, T^* w\rangle \\ &= \langle Tu, w\rangle - \langle Tu, w\rangle \\ &= 0\end{aligned}$$

Since $\alpha \neq \beta$, $\langle u, w\rangle = 0$ must hold.

7.23 *T is normal $\iff$ the real and imaginary parts of T commute*

Suppose $\mathbf{F} = \mathbf{C}$ and $T \in \mathcal{L}(V)$. Then T is normal if and only if there exist commuting self-adjoint operators A and B such that $T = A + iB$.

Proof First, suppose T is normal. Then, let

$A = \dfrac{T + T^*}{2}$, $B = \dfrac{T - T^*}{2i}$ & notice $A + iB = \dfrac{T+T^*}{2} + \dfrac{T - T^*}{2} = \dfrac{2T}{2} = T$.

Since $A^* = \dfrac{T^* + T}{2} = \dfrac{T + T^*}{2} = A$ & $B^* = \overline{\left(\dfrac{1}{2i}\right)}(T^* - T) = \left(\dfrac{-1}{2i}\right)(T^* - T) = \dfrac{T - T^*}{2i} = B$,

both A & B are self-adjoint. Moreover,

$$AB = \left(\frac{T^* + T}{2}\right)\left(\frac{T - T^*}{2i}\right) = \left(\frac{1}{4i}\right)\left(T^*T - T^*T^* + TT - TT^*\right) = \frac{TT - T^*T^*}{4i}$$

$$BA = \left(\frac{T - T^*}{2i}\right)\left(\frac{T + T^*}{2}\right) = \left(\frac{1}{4i}\right)\left(TT + TT^* - T^*T - T^*T^*\right) = \frac{TT - T^*T^*}{4i}$$

So, we are done!

Now, in the other direction, if $T = A + iB$ for A, B self adjoint commuting operators, then $T^* = A^* - iB^* = A - iB$. So,

$T^*T = (A - iB)(A + iB) = A^2 + iAB - iBA + B^2 = A^2 + B^2$

$TT^* = (A + iB)(A - iB) = A^2 - iAB + iBA + B^2 = A^2 + B^2$ & T is normal.