Adjoints

7.1 definition: *adjoint*, T^*

Suppose $T \in \mathcal{L}(V, W)$. The *adjoint* of T is the function $T^* : W \to V$ such that

$$\langle Tv, w\rangle_W = \langle v, T^*w\rangle_V$$

for every $v \in V$ and every $w \in W$.

$\boxed{\text{Idea}}$ How do we know such a T^* exists? Consider the linear functionals $\varphi_w \in W'$ given by $\varphi_w(u) = \langle u, w\rangle$. Then, $\varphi_w \circ T \in V'$, since the composition of linear maps is linear. Hence, by the Riesz rep theorem, there is some $v_w \in V$ s.t. $(\varphi_w \circ T)(v) = \langle v, v_w\rangle$, which says $\langle Tv, w\rangle = \langle v, v_w\rangle$. So, $T^*w = v_w$ is how we define T^*. Now, is it true that $T^* \in \mathcal{L}(W, V)$? Yes!

7.4 *adjoint of a linear map is a linear map*

If $T \in \mathcal{L}(V, W)$, then $T^* \in \mathcal{L}(W, V)$.

$\boxed{\text{Proof}}$ Suppose $T \in \mathcal{L}(V, W)$. Now, if $v \in V$ & $w_1, w_2 \in W$, then

$$\begin{aligned}\langle Tv, w_1 + w_2\rangle &= \langle Tv, w_1\rangle + \langle Tv, w_2\rangle \\ &= \langle v, T^*w_1\rangle + \langle v, T^*w_2\rangle \\ &= \langle v, T^*w_1 + T^*w_2\rangle\end{aligned}$$

So, $\langle Tv, w_1 + w_2\rangle = \langle v, T^*(w_1 + w_2)\rangle = \langle v, T^*w_1 + T^*w_2\rangle$ and we may conclude (via the uniqueness ensured by Riesz rep) that $T^*(w_1 + w_2) = T^*w_1 + T^*w_2$.

We do the same for homogeneity, taking $\lambda \in \mathbb{F}$ & noting that

$$\langle v, T^*(\lambda w)\rangle = \langle Tv, \lambda w\rangle = \bar{\lambda}\langle Tv, w\rangle = \bar{\lambda}\langle v, T^*w\rangle = \langle v, \lambda(T^*w)\rangle$$

So, $T^*(\lambda w) = \lambda T^*w$ & T^* is linear!

7.5 *properties of the adjoint*

Suppose $T \in \mathcal{L}(V, W)$. Then

(a) $(S+T)^* = S^* + T^*$ for all $S \in \mathcal{L}(V, W)$; ✓

(b) $(\lambda T)^* = \bar{\lambda} T^*$ for all $\lambda \in \mathbf{F}$; ✓

(c) $(T^*)^* = T$; ✓

(d) $(ST)^* = T^* S^*$ for all $S \in \mathcal{L}(W, U)$ (here U is a finite-dimensional inner product space over $\mathbf{F}$); ✓

(e) $I^* = I$, where I is the identity operator on V; ✓

(f) if T is invertible, then T^* is invertible and $(T^*)^{-1} = (T^{-1})^*$. ✓

$\boxed{\text{Proof}}$

(a) $$\underbrace{\langle (S+T)v, w\rangle}_{= \langle v, (S+T)^* w\rangle} = \langle Sv + Tv, w\rangle = \langle Sv, w\rangle + \langle Tv, w\rangle = \langle v, S^* w\rangle + \langle v, T^* w\rangle = \langle v, S^* w + T^* w\rangle = \langle v, (S^* + T^*) w\rangle$$

So, $(S+T)^* = S^* + T^*$.

(b) $$\langle v, (\lambda T)^* w\rangle = \langle (\lambda T)v, w\rangle = \lambda \langle Tv, w\rangle = \lambda \langle v, T^* w\rangle = \langle v, \bar{\lambda} T^* w\rangle$$ So, $(\lambda T)^* = \bar{\lambda} T^*$

(c) $$\langle Tv, w\rangle = \langle v, T^* w\rangle = \overline{\langle T^* w, v\rangle} = \overline{\langle w, (T^*)^* v\rangle}, \langle (T^*)^* v, w\rangle$$

Hence, $Tv = (T^*)^* v$ for all $v \in U$, i.e. $T = (T^*)^*$

(d) $$\langle v, (ST)^* w\rangle = \langle STv, w\rangle = \langle Tv, S^* w\rangle = \langle v, T^* S^* w\rangle$$

So, $(ST)^* = T^* S^*$

(e) $$\langle u, I^* v\rangle = \langle Iu, v\rangle = \langle u, v\rangle = \langle u, Iv\rangle$$

So, $I^* = I$

(f) ∵ T is invertible, Then, $T^{-1} T = I$ & $(T^{-1}T)^* = T^* (T^{-1})^* = I^* = I$

same!

Hence, T^* is invertible & $(T^*)^{-1} = (T^{-1})^*$

(Technically, we should also note that $TT^{-1} = I$ & $(TT^{-1})^* = (T^{-1})^* T^* = I^* = I$ as well)

7.6 *null space and range of T^**

Suppose $T \in \mathcal{L}(V, W)$. Then

(a) $\operatorname{null} T^* = (\operatorname{range} T)^\perp$;

(b) $\operatorname{range} T^* = (\operatorname{null} T)^\perp$;

(c) $\operatorname{null} T = (\operatorname{range} T^*)^\perp$;

(d) $\operatorname{range} T = (\operatorname{null} T^*)^\perp$.

$\boxed{\text{Proof}}$ (a) $w \in \operatorname{null} T^* \iff T^* w = 0$

$\iff \langle v, T^* w \rangle = 0$ for all $v \in V$

$\iff \langle Tv, w \rangle = 0$ for all $v \in V$

$\iff w \in (\operatorname{range} T)^\perp$ (since it is orthogonal to every Tv)

(b) This is part (d) w/ T replaced by T^*, since $(T^*)^* = T$

(c) This is part (a) w/ T replaced by T^*, since $(T^*)^* = T$

(d) Using (a), $\operatorname{null} T^* = (\operatorname{range} T)^\perp$, so $(\operatorname{null} T^*)^\perp = ((\operatorname{range} T)^\perp)^\perp = \operatorname{range} T$.

7.7 definition: *conjugate transpose, A^**

The $\boxed{\textit{conjugate transpose}}$ of an m-by-n matrix A is the n-by-m matrix A^* obtained by interchanging the rows and columns and then taking the complex conjugate of each entry. In other words, if $j \in \{1, \ldots, n\}$ and $k \in \{1, \ldots, m\}$, then

$$(A^*)_{j,k} = \overline{A_{k,j}}.$$

7.9 *matrix of T^* equals conjugate transpose of matrix of T*

Let $T \in \mathcal{L}(V, W)$. Suppose $e_1, \ldots, e_n$ is an orthonormal basis of V and $f_1, \ldots, f_m$ is an orthonormal basis of W. Then $\mathcal{M}(T^*, (f_1, \ldots, f_m), (e_1, \ldots, e_n))$ is the conjugate transpose of $\mathcal{M}(T, (e_1, \ldots, e_n), (f_1, \ldots, f_m))$. In other words,

$$\mathcal{M}(T^*) = (\mathcal{M}(T))^*.$$

$\boxed{\text{Proof}}$ (sketch)

$$Te_k = \langle Te_k, f_1 \rangle f_1 + \cdots + \langle Te_k, f_m \rangle f_m$$

So, $\mathcal{M}(T)_{j,k} = \langle Te_k, f_j \rangle = \langle e_k, T^* f_j \rangle = \overline{\langle T^* f_j, e_k \rangle} = \overline{\mathcal{M}(T^*)_{k,j}}$

for the same reason, i.e.,

$$T^* f_j = \langle T^* f_j, e_1 \rangle e_1 + \cdots + \langle T^* f_j, e_n \rangle e_n$$

Self-Adjoint Operators

7.10 definition: *self-adjoint*

An operator $T \in \mathcal{L}(V)$ is called *self-adjoint* if $T = T^*$.

An operator $T \in \mathcal{L}(V)$ is self-adjoint if and only if

$$\langle Tv, w\rangle = \langle v, Tw\rangle \checkmark$$

for all $v, w \in V$.

i.e. $\langle Tv,w\rangle - \langle v,Tw\rangle = 0$

$\langle v, T^*w\rangle - \langle v,Tw\rangle = 0$

$\langle v, (T^*-T)w\rangle = 0$

$T^*-T=0$

$T=T^*$

$\langle (T^*-T)v, w\rangle = 0$

$\langle (T-T^*)v, w\rangle = 0$

So, $\langle (T-T^*)v, v\rangle = 0$

$\Rightarrow T-T^*=0$

or $T=T^*$

Notice, if $T \in \mathcal{L}(V,W)$, then $T^* \in \mathcal{L}(W,V)$

So $T=T^*$ requires $V=W$

7.12 *eigenvalues of self-adjoint operators*

Every eigenvalue of a self-adjoint operator is real.

$\boxed{\text{Proof}}$ Suppose $T \in \mathcal{L}(V)$ is self-adjoint. Let λ be an eigenvalue of T, and let $v \in V$ be a corresponding eigenvector (which means $v \neq 0$). Then,

$$\lambda\|v\|^2 = \lambda\langle v,v\rangle = \langle \lambda v, v\rangle = \langle Tv, v\rangle = \langle v, Tv\rangle = \langle v, \lambda v\rangle = \bar{\lambda}\langle v,v\rangle = \bar{\lambda}\|v\|^2$$

So, $\lambda\|v\|^2 = \bar{\lambda}\|v\|^2$, and dividing by $\|v\|^2 \neq 0$, we get $\lambda = \bar{\lambda}$.

So, λ is real!

7.13 *Tv is orthogonal to v for all v* $\iff$ $T = 0$ (*assuming* $\mathbf{F} = \mathbf{C}$)

Suppose V is a complex inner product space and $T \in \mathcal{L}(V)$. Then

$$\langle Tv, v\rangle = 0 \text{ for every } v \in V \iff T = 0.$$

The RHS is of the form $\langle Tv,v\rangle$ in each term, so $\langle Tu,w\rangle = 0$ $\forall u,w \in V$. Hence, $Tu=0$ $\Rightarrow T=0$.

$\boxed{\text{proof}}$ If $u, w \in V$, then

$$\boxed{\langle Tu,w\rangle = \frac{\langle T(u+w), u+w\rangle - \langle T(u-w), u-w\rangle}{4} + \frac{\langle T(u+iw), u+iw\rangle - \langle T(u-iw), u-iw\rangle}{4}\, i}$$

$$= \frac{\langle Tu+Tw, u+w\rangle - \langle Tu-Tw, u-w\rangle}{4} + \frac{\langle Tu+iTw, u+iw\rangle - \langle Tu-iTw, u-iw\rangle}{4}\, i$$

$$= \frac{\cancel{\langle Tu,u\rangle} + \langle Tu,w\rangle + \cancel{\langle Tw,u\rangle} + \cancel{\langle Tw,w\rangle} - \cancel{\langle Tu,u\rangle} + \langle Tu,w\rangle + \cancel{\langle Tw,u\rangle} - \cancel{\langle Tw,w\rangle}}{4}$$

$$+ \frac{\cancel{\langle Tu,u\rangle} + \langle Tu,w\rangle - \cancel{\langle Tw,u\rangle} + \cancel{\langle Tw,w\rangle} - \cancel{\langle Tu,u\rangle} + \langle Tu,w\rangle - \cancel{\langle Tw,u\rangle} - \cancel{\langle Tw,w\rangle}}{4}$$

$$= \frac{4\langle Tu,w\rangle}{4} = \langle Tu,w\rangle$$