

6C Orthogonal Complements and Minimization Problems

6.46 definition: orthogonal complement, $U^\perp$

If U is a subset of V , then the orthogonal complement of U , denoted by $U^\perp$, is the set of all vectors in V that are orthogonal to every vector in U :

$$U^\perp = \{v \in V : \langle u, v \rangle = 0 \text{ for every } u \in U\}.$$



Since $\langle \cdot, v \rangle$ is a linear functional on V , all we are doing is taking $v \in V$ s.t. $U \subseteq \text{null}(\langle \cdot, v \rangle)$

6.48 properties of orthogonal complement

- (a) If U is a subset of V , then $U^\perp$ is a subspace of V .
- (b) $\{0\}^\perp = V$.
- (c) $V^\perp = \{0\}$.
- (d) If U is a subset of V , then $U \cap U^\perp \subseteq \{0\}$.
- (e) If G and H are subsets of V and $G \subseteq H$, then $H^\perp \subseteq G^\perp$.

Proof (a) If $v \in U^\perp$, then $\langle u, v \rangle = 0$ for all $u \in U$.

So, $\langle u, \lambda v \rangle = \lambda \langle u, v \rangle = \lambda(0) = 0$ for all $u \in U$ and $\lambda v \in U^\perp$ as well for any $\lambda \in \mathbb{F}$. Similarly, if $w \in U^\perp$, then $\langle u, v+w \rangle = \langle u, v \rangle + \langle u, w \rangle = 0 + 0 = 0$. So, $v+w \in U^\perp$ as well. Clearly, $\langle u, 0 \rangle = 0$, so $0 \in U^\perp$. Hence, $U^\perp \leq V$.

(b) $\{0\}^\perp = \{v \in V : \langle 0, v \rangle = 0\} = V$ (by definition of $\langle \cdot, \cdot \rangle$)

(c) $V^\perp = \{v \in V : \langle v, v \rangle = 0 \text{ for all } v \in V\} \subseteq \{v \in V : \langle v, v \rangle = 0\} = \{v \in V : \|v\|^2 = 0\} = \{0\}$

(d) $U \cap U^\perp = U \cap \{v \in V : \langle u, v \rangle = 0 \text{ for all } u \in U\} = \{v \in U : \langle u, v \rangle = 0 \text{ for all } u \in U\} \subseteq \{v \in U : \langle v, v \rangle = 0\} = \{0\}$.

(e) If $G \subseteq H$, then $H^\perp = \{v \in V : \langle h, v \rangle = 0 \text{ for all } h \in H\} \subseteq \{v \in V : \langle g, v \rangle = 0 \text{ for all } g \in G\} = G^\perp$

6.49 direct sum of a subspace and its orthogonal complement

Suppose U is a finite-dimensional subspace of V . Then

$$V = U \oplus U^\perp.$$

Proof

Let $v \in V$ & $\{e_1, \dots, e_n\}$ is an ONB of U . Then,

$$v = v + 0 = v + (\langle v, e_1 \rangle e_1 + \dots + \langle v, e_n \rangle e_n) - (\langle v, e_1 \rangle e_1 + \dots + \langle v, e_n \rangle e_n) \quad \text{P}_U v$$

Let $w = v - (\langle v, e_1 \rangle e_1 + \dots + \langle v, e_n \rangle e_n)$ & let $u = \langle v, e_1 \rangle e_1 + \dots + \langle v, e_n \rangle e_n$

So that $v = u + w$. Certainly, $u \in U$, since it is a linear combo of a basis for U . Moreover,

$$\begin{aligned} \langle w, e_k \rangle &= \langle v - u, e_k \rangle \\ &= \langle v, e_k \rangle - \langle u, e_k \rangle \\ &= \langle v, e_k \rangle - \langle \langle v, e_1 \rangle e_1 + \dots + \langle v, e_n \rangle e_n, e_k \rangle \\ &= \langle v, e_k \rangle - (\langle v, e_1 \rangle \langle e_1, e_k \rangle + \dots + \langle v, e_n \rangle \langle e_n, e_k \rangle) \\ &= \langle v, e_k \rangle - (\langle v, e_k \rangle \langle e_k, e_k \rangle) \\ &= \langle v, e_k \rangle - \langle v, e_k \rangle \\ &= 0 \end{aligned}$$

So, $w \in U^\perp$, since w is orthogonal to every element in a basis of U . Hence, $V = U + U^\perp$. Moreover, this sum is direct since

$U \cap U^\perp = \{0\}$ by the previous theorem.

6.51 dimension of orthogonal complement

Suppose V is finite-dimensional and U is a subspace of V . Then

$$\dim U^\perp = \dim V - \dim U.$$

6.52 orthogonal complement of the orthogonal complement

Suppose U is a finite-dimensional subspace of V . Then

$$U = (U^\perp)^\perp.$$

Proof

Notice $U^\perp = \{v \in V : \langle u, v \rangle = 0 \ \forall u \in U\}$

So, $(U^\perp)^\perp = \{v \in V : \langle w, v \rangle = 0 \ \forall w \in U^\perp\}$

but notice that if $u \in U$, then $\langle w, u \rangle = \overline{\langle u, w \rangle} = \overline{0} = 0$ whenever $w \in U^\perp$. So, $U \subseteq (U^\perp)^\perp$.

For the reverse inclusion, $\$ v \in (U^\perp)^\perp$. By the previous result, $v = u + w$ where $u \in U$ & $w \in U^\perp$. So, $\underline{v - u = w \in U^\perp}$. So, $v - u \in U^\perp$, and since $U \subseteq (U^\perp)^\perp$, $\underline{v - u \in (U^\perp)^\perp}$. Hence, $v - u \in U^\perp \cap (U^\perp)^\perp = \{0\}$, hence $v - u = 0$ & $v = u \in U$. So $(U^\perp)^\perp \subseteq U$. Therefore, $U = (U^\perp)^\perp$.

6.54 $U^\perp = \{0\} \iff U = V$ (for U a finite-dimensional subspace of V)

Suppose U is a finite-dimensional subspace of V . Then

$$U^\perp = \{0\} \iff U = V.$$

Proof

$$U^\perp = \{0\} \iff (U^\perp)^\perp = \{0\}^\perp$$

$$\iff U = V \quad \left(\begin{array}{l} \text{since } (U^\perp)^\perp = U \\ \& \{0\}^\perp = V \end{array} \right)$$

6.55 definition: *orthogonal projection*, P_U

Suppose U is a finite-dimensional subspace of V . The orthogonal projection of V onto U is the operator $P_U \in \mathcal{L}(V)$ defined as follows: For each $v \in V$, write $v = u + w$, where $u \in U$ and $w \in U^\perp$. Then let $P_U v = u$.

For an ONB $e_1, \dots, e_n$ of U , $P_U v$ is just $P_U v = \langle v, e_1 \rangle e_1 + \dots + \langle v, e_n \rangle e_n$

6.57 *properties of orthogonal projection* P_U

Suppose U is a finite-dimensional subspace of V . Then

- (a) $P_U \in \mathcal{L}(V)$; $\leftarrow$ easy!
- (b) $P_U u = u$ for every $u \in U$; $\leftarrow$ directly from this
- (c) $P_U w = 0$ for every $w \in U^\perp$; $\leftarrow$ easy, from definition of P_U
- (d) $\text{range } P_U = U$; Use (b) & (c) w/ $v = u \oplus u^\perp$
- (e) $\text{null } P_U = U^\perp$; Same logic!
- (f) $v - P_U v \in U^\perp$ for every $v \in V$; $\leftarrow v = u + w, u \in U, w \in U^\perp$
- (g) $P_U^2 = P_U$; If $v = u + w, u \in U, w \in U^\perp$
 $P_U v = u$ & $P_U^2 v = P_U u = u$ $\leftarrow v = P_U v + w \Rightarrow v - P_U v = w \in U^\perp$
- (h) $\|P_U v\| \leq \|v\|$ for every $v \in V$; $\|u\| \leq \|u + w\| = \|v\|$ since u & w are orthogonal
- (i) if $e_1, \dots, e_m$ is an orthonormal basis of U and $v \in V$, then

$$P_U v = \langle v, e_1 \rangle e_1 + \dots + \langle v, e_m \rangle e_m.$$

Proof

(h) Since $V = U \oplus U^\perp$, write $v \in V$ as $v = u + w$ where $u \in U$ & $w \in U^\perp$. Since $P_U v = u$, and since u & w are orthogonal, we get $\|P_U v\|^2 = \|u\|^2 \leq \|u\|^2 + \|w\|^2 = \|u + w\|^2$ via the Pythagorean theorem. Taking a square root, $\|P_U v\| \leq \|u + w\| = \|v\|$.

(i) If $u \in U$, then $u = \langle u, e_1 \rangle e_1 + \dots + \langle u, e_m \rangle e_m$ for any ONB $e_1, \dots, e_m$ of U by a previous result. So, since

$$P_U v = P_U(u + w) = u, \text{ we get that } P_U v = \underline{\langle u, e_1 \rangle e_1 + \dots + \langle u, e_m \rangle e_m}$$

But notice: since $w \in U^\perp$, $\langle w, e_k \rangle = 0$, so

$$\begin{aligned} P_U v &= \langle u, e_1 \rangle e_1 + \dots + \langle u, e_m \rangle e_m \\ &= (\langle u, e_1 \rangle + 0) e_1 + \dots + (\langle u, e_m \rangle + 0) e_m \\ &= (\langle u, e_1 \rangle + \langle w, e_1 \rangle) e_1 + \dots + (\langle u, e_m \rangle + \langle w, e_m \rangle) e_m \\ &= \langle v, e_1 \rangle e_1 + \dots + \langle v, e_m \rangle e_m. \end{aligned}$$

6.58 Riesz representation theorem, revisited

Suppose V is finite-dimensional. For each $v \in V$, define $\varphi_v \in V'$ by

$$\varphi_v(u) = \langle u, v \rangle$$

for each $u \in V$. Then $v \mapsto \varphi_v$ is a one-to-one function from V onto V' .

Note If $F = \mathbb{R}$, then $\varphi_{\lambda v}(u) = \langle u, \lambda v \rangle = \bar{\lambda} \langle u, v \rangle = \lambda \langle u, v \rangle = \lambda \varphi_v(u)$

So, in the real case, the map $T(v) = \varphi_v$ is linear! i.e.

$T \in \mathcal{L}(V, V')$ if V is a real inner product space. But, if $F = \mathbb{C}$, then T is not linear.

Proof (if $F = \mathbb{R}$) Let $T \in \mathcal{L}(V, V')$ be $Tv = \varphi_v$. Then,

Notice that $\text{null } T = \{v \in V : Tv = 0 \in V'\} = \{v \in V : \langle u, v \rangle = 0 \forall u \in V\} = \underbrace{\{v \in V : \langle v, v \rangle = 0\}}_{\{0\}}$

Hence, $\text{null } T = \{0\}$ & T is injective. Since $\dim V = \dim V'$, T is then also surjective & an isomorphism.

(If $F = \mathbb{C}$) In this case, the above works to show that the set of vectors $v \in V$ s.t. $\varphi_v = 0$ is just $\{0\}$.

For surjectivity, all we need to do is note that $\varphi_0 = 0$ & for $v \in V, v \neq 0$, $\text{null } \varphi_v$ has dimension $\dim V - 1$. So, $(\text{null } \varphi_v)^\perp$ is one dimensional. The same is true of any $\varphi \in V'$, so all we need to do is "match up" the null spaces' orthogonal complements & scale appropriately. For details, see page 216 & 217 in the book.