Linear Functionals on Inner Product Spaces

$\boxed{\text{Idea}}$ If we take $v \in V$, we can obtain a linear functional $\varphi \in V'$ via $\varphi(u) = \langle u, v\rangle$. So, the mapping $v \mapsto \langle \cdot, v\rangle$ sends V to some portion of V'. But how much? What liner functionals are of this form?

$\boxed{\text{Ex}}$ Let $\varphi \in \mathcal{L}(\mathbb{F}^3, \mathbb{F}) = (\mathbb{F}^3)'$ be $\varphi(x_1, x_2, x_3) = 2x_1 - 5x_2 + x_3$.

Then, $\varphi(x) = \langle x, v\rangle$ for $v = (2, -5, 1)$ (if $\langle \cdot, \cdot \rangle$ is the dot product).

But what about, say, $\varphi \in (\mathcal{P}_n(\mathbb{R}))'$ w/ $\varphi(p) = \int_a^b p(t)\cos(t)\,dt$? $\leftarrow$???

6.42 *Riesz representation theorem*

Suppose V is finite-dimensional and φ is a linear functional on V. Then there is a unique vector $v \in V$ such that

i.e. $\varphi \in V'$

$$\varphi(u) = \langle u, v\rangle$$

for every $u \in V$.

$\boxed{\text{Proof}}$ Let $e_1, \dots, e_n$ be an ONB of V. Then, if $\varphi \in V'$,

$$\begin{aligned}
\varphi(u) &= \varphi(\langle u, e_1\rangle e_1 + \dots + \langle u, e_n\rangle e_n) \quad (\text{since } u = \langle u, e_1\rangle e_1 + \dots + \langle u, e_n\rangle e_n)\\
&= \langle u, e_1\rangle \overline{\overline{\varphi(e_1)}} + \dots + \langle u, e_n\rangle \overline{\overline{\varphi(e_n)}} \quad (\text{via linearity})\\
&= \langle u, \overline{\varphi(e_1)}\, e_1\rangle + \dots + \langle u, \overline{\varphi(e_n)}\, e_n\rangle \quad (\text{via } \langle x, \lambda y\rangle = \bar{\lambda}\langle x, y\rangle)\\
&= \langle u, \overline{\varphi(e_1)}\, e_1 + \dots + \overline{\varphi(e_n)}\, e_n\rangle \quad (\text{via additivity in 2}^{\text{nd}} \text{ slot})\\
&= \langle u, v\rangle \quad \text{for } v = \overline{\varphi(e_1)}\, e_1 + \dots + \overline{\varphi(e_n)}\, e_n
\end{aligned}$$

So, there does exist some $v \in V$ s.t. $\varphi(u) = \langle u, v\rangle$ for all $u \in U$. For uniqueness, suppose $v_1, v_2 \in V$ are both s.t. $\varphi(u) = \langle u, v_1\rangle$ & $\varphi(u) = \langle u, v_2\rangle$. Then,

$$\begin{aligned}
0 &= \varphi(u) - \varphi(u)\\
&= \langle u, v_1\rangle - \langle u, v_2\rangle\\
&= \langle u, v_1 - v_2\rangle
\end{aligned}$$

So, taking $u = v_1 - v_2$, we get $0 = \langle v_1 - v_2, v_1 - v_2\rangle = \|v_1 - v_2\|^2$.

Hence, $v_1 - v_2 = 0$ & $v_1 = v_2$. So, this vector is unique!

Ex Is there a poly. $q(t)$ s.t. $\int_{-1}^{1} p(t)\cos(\pi t)\,dt = \int_{-1}^{1} p(t)q(t)\,dt$ for all $p \in \mathcal{P}_2(\mathbb{R})$? On first glance, this seems strange to imagine, since $\cos(\pi t)$ is not a polynomial! Nonetheless, there is!

Define an inner product on $\mathcal{P}_2(\mathbb{R})$ via $\langle p,q\rangle = \int_{-1}^{1} p(t)q(t)\,dt$.
Since $\varphi(p) = \int_{-1}^{1} p(t)\cos(\pi t)\,dt$ is a linear functional on $\mathcal{P}_2(\mathbb{R})$, the Riesz rep. theorem says that there does exist $q \in \mathcal{P}_2(\mathbb{R})$ s.t. $\varphi(p) = \langle p,q\rangle$. To get it, we need an ONB of $\mathcal{P}_2(\mathbb{R})$, call it e_1, e_2, e_3, so we can construct $\boxed{q = \overline{\varphi(e_1)}e_1 + \overline{\varphi(e_2)}e_2 + \overline{\varphi(e_3)}e_3}$, since

$$\begin{aligned}
\langle p,q\rangle &= \langle p, \overline{\varphi(e_1)}e_1 + \overline{\varphi(e_2)}e_2 + \overline{\varphi(e_3)}e_3\rangle \\
&= \langle p,e_1\rangle\varphi(e_1) + \langle p,e_2\rangle\varphi(e_2) + \langle p,e_3\rangle\varphi(e_3) \\
&= \varphi(\langle p,e_1\rangle e_1 + \langle p,e_2\rangle e_2 + \langle p,e_3\rangle e_3) \\
&= \varphi(p)
\end{aligned}$$

To get this ONB, let's start w/ the basis $v_1 = 1, v_2 = x, v_3 = x^2$ then apply the GS process. First, take $f_1 = v_1 = 1$. Thus,
$\|f_1\|^2 = \int_{-1}^{1} (1)^2\,dt = 2$. So, $f_2 = v_2 - \frac{\langle v_2, f_1\rangle}{\|f_1\|^2} f_1 = x - \frac{\langle x,1\rangle}{2}1 = x - \cancel{\frac{\int_{-1}^{1}(t)(1)\,dt}{2}}\,1$

hence, $f_2 = x$. So, $\|f_2\|^2 = \int_{-1}^{1} (t)^2\,dt = \frac{2}{3}$. So

$$f_3 = v_3 - \frac{\langle v_3, f_1\rangle}{\|f_1\|^2} - \frac{\langle v_3, f_2\rangle}{\|f_2\|^2} f_2 = x^2 - \frac{\langle x^2, 1\rangle}{2}1 - \frac{\langle x^2, x\rangle}{(2/3)}x = x^2 - \frac{1}{3}$$

Hence, $\|f_3\|^2 = \int_{-1}^{1}\left(t^2 - \frac{1}{3}\right)^2 dt = \frac{8}{45}$. So, the list e_1, e_2, e_3 w/ $e_i = \frac{f_i}{\|f_i\|}$ is an ONB & $e_1 = \sqrt{1/2}$, $e_2 = \left(\sqrt{\frac{3}{2}}\right)x$, $e_3 = \left(\sqrt{\frac{45}{8}}\right)\left(x^2 - \frac{1}{3}\right)$

Hence, the poly.
$$\begin{aligned}
q(x) &= \overbrace{\left(\int_{-1}^{1}\left(\sqrt{\tfrac{1}{2}}\right)\cos(\pi t)\,dt\right)}^{\varphi(e_1)}\overbrace{\sqrt{\tfrac{1}{2}}}^{e_1} + \overbrace{\left(\int_{-1}^{1}\left(\sqrt{\tfrac{3}{2}}\right)t\cos(\pi t)\,dt\right)}^{\varphi(e_2)}\overbrace{\sqrt{\tfrac{3}{2}}\,x}^{e_2} \\
&\quad + \overbrace{\left(\int_{-1}^{1}\sqrt{\tfrac{45}{8}}\left(t^2 - \tfrac{1}{3}\right)\cos(\pi t)\,dt\right)}^{\varphi(e_3)}\overbrace{\sqrt{\tfrac{45}{8}}\left(x^2 - \tfrac{1}{3}\right)}^{e_3} \\
&= \frac{15}{2\pi^2}\left(1 - 3x^2\right)
\end{aligned}$$

$\boxed{\text{Ex}}$ (continued)

What about if we want to do this for a poly of a different degree? All we have to do is repeat this process with an ONB of $\mathcal{P}_n(\mathbb{R})$ where n is our desired degree.

For example, if $n=5$, then the poly.

$$q(x) = \frac{105}{8\pi^4}\left((27-2\pi^2) + (24\pi^2-270)x^2 + (315-30\pi^2)x^4\right)$$

Satisfies $\displaystyle\int_{-1}^{1} p(t)\cos(\pi t)\,dt = \int_{-1}^{1} p(t)q(t)\,dt$

for all $p \in \mathcal{P}_5(\mathbb{R})$.