6B Orthonormal Bases

Orthonormal Lists and the Gram–Schmidt Procedure

6.22 definition: *orthonormal*

ortho — orthogonal, pairwise
normal — "normalized" i.e. norm 1

- A list of vectors is called *orthonormal* if each vector in the list has norm 1 and is orthogonal to all the other vectors in the list.
- In other words, a list $e_1, \ldots, e_m$ of vectors in V is orthonormal if

$$\langle e_j, e_k \rangle = \begin{cases} 1 & \text{if } j = k, \\ 0 & \text{if } j \neq k \end{cases}$$

so, $\langle e_j, e_j \rangle = 1 = \|e_j\|^2$ & $\|e_j\| = 1$

for all $j, k \in \{1, \ldots, m\}$.

6.24 *norm of an orthonormal linear combination*

Suppose $e_1, \ldots, e_m$ is an orthonormal list of vectors in V. Then

$$\|a_1 e_1 + \cdots + a_m e_m\|^2 = |a_1|^2 + \cdots + |a_m|^2$$

for all $a_1, \ldots, a_m \in \mathbf{F}$.

$\boxed{\text{Proof}}$ Using the pythagorean theorem (which, recall, says that $\|v+w\|^2 = \|v\|^2 + \|w\|^2$ for v, w orthogonal), we get

$$\begin{aligned} \|a_1e_1 + \cdots + a_m e_m\|^2 &= \|a_1 e_1\|^2 + \|a_2 e_2 + \cdots + a_m e_m\|^2 \\ &\;\vdots \\ &= \|a_1 e_1\|^2 + \cdots + \|a_m e_m\|^2 \\ &= |a_1|^2 \|e_1\|^2 + \cdots + |a_m|^2 \|e_m\|^2 \\ &= |a_1|^2 + \cdots + |a_m|^2 \end{aligned}$$

Since $\langle a_1 e_1, a_2 e_2 + \cdots + a_m e_m \rangle = a_1(a_2 \langle e_1, e_2 \rangle + \cdots + a_m \langle e_1, e_m \rangle) = a_1(a_2(0) + \cdots + a_m(0)) = 0$

6.25 *orthonormal lists are linearly independent*

Every orthonormal list of vectors is linearly independent.

$\boxed{\text{Proof}}$ Let $e_1, \ldots, e_m$ be an ON list in V. Let $a_1, \ldots, a_m \in \mathbf{F}$ be a list of scalars s.t. $a_1 e_1 + \cdots + a_m e_m = 0$. Then, using the previous result, $0 = \|a_1 e_1 + \cdots + a_m e_m\|^2 = |a_1|^2 + \cdots + |a_m|^2$.

So, $a_1 = \cdots = a_m = 0$ and $e_1, \ldots, e_m$ is lin. indep.

6.26 *Bessel's inequality*

Suppose $e_1, \dots, e_m$ is an orthonormal list of vectors in V. If $v \in V$ then

$$|\langle v, e_1\rangle|^2 + \cdots + |\langle v, e_m\rangle|^2 \le \|v\|^2.$$

$\boxed{\text{Proof}}$ Set $u = \langle v, e_1\rangle e_1 + \dots + \langle v, e_m\rangle e_m$. Let $w = v - u$. Then, $v = u - u + v = u + w$. Now, if $\langle w, u\rangle = 0$, then by the Pythagorean theorem, we have $\|v\|^2 = \|u\|^2 + \|w\|^2 \ge \|u\|^2 = |\langle v, e_1\rangle|^2 + \dots + |\langle v, e_m\rangle|^2$ by the previous result. So, we need only show $\langle w, u\rangle = 0$. To wit,

$$\begin{aligned}
\langle w, e_k\rangle = \langle v - u, e_k\rangle &= \langle v - (\langle v, e_1\rangle e_1 + \dots + \langle v, e_m\rangle e_m), e_k\rangle \\
&= \langle v, e_k\rangle - (\langle v, e_1\rangle\langle e_1, e_k\rangle + \dots + \langle v, e_m\rangle\langle e_m, e_k\rangle) \\
&= \langle v, e_k\rangle - \langle v, e_k\rangle\langle e_k, e_k\rangle \\
&= \langle v, e_k\rangle - \langle v, e_k\rangle\|e_k\|^2 \\
&= \langle v, e_k\rangle - \langle v, e_k\rangle \\
&= 0
\end{aligned}$$

Hence,
$$\begin{aligned}
\langle w, u\rangle &= \langle w, \langle v, e_1\rangle e_1 + \dots + \langle v, e_m\rangle e_m\rangle \\
&= \overline{\langle v, e_1\rangle}\langle w, e_1\rangle + \dots + \overline{\langle v, e_m\rangle}\langle w, e_m\rangle \\
&= \overline{\langle v, e_1\rangle}(0) + \dots + \overline{\langle v, e_m\rangle}(0) \\
&= 0.
\end{aligned}$$

6.27 definition: *orthonormal basis*

An *orthonormal basis* of V is an orthonormal list of vectors in V that is also a basis of V.

6.28 *orthonormal lists of the right length are orthonormal bases*

Suppose V is finite-dimensional. Then every orthonormal list of vectors in V of length $\dim V$ is an orthonormal basis of V.

$\boxed{\text{Proof}}$ If $e_1, \dots, e_{\dim V}$ is orthonormal, then it is a lin. indep. list of length $\dim V$ & is therefore a basis.

6.30 *writing a vector as a linear combination of an orthonormal basis*

Suppose $e_1, \ldots, e_n$ is an orthonormal basis of V and $u, v \in V$. Then

(a) $v = \langle v, e_1\rangle e_1 + \cdots + \langle v, e_n\rangle e_n$, ✓

(b) $\|v\|^2 = |\langle v, e_1\rangle|^2 + \cdots + |\langle v, e_n\rangle|^2$,

(c) $\langle u, v\rangle = \langle u, e_1\rangle\overline{\langle v, e_1\rangle} + \cdots + \langle u, e_n\rangle\overline{\langle v, e_n\rangle}$.

Proof (a) Since $e_1, \ldots, e_n$ is an ONB, the exist scalers $a_1, \ldots, a_n \in \mathbb{F}$ s.t. $v = a_1e_1 + \cdots + a_ne_n$. But, notice:

$$\begin{aligned}\langle v, e_k\rangle &= \langle a_1e_1 + \cdots + a_ne_n, e_k\rangle \\ &= a_1\langle e_1, e_k\rangle + \cdots + a_n\langle e_n, e_k\rangle \\ &= a_k\langle e_k, e_k\rangle \\ &= a_k\end{aligned}$$

Hence, $v = \langle v, e_1\rangle e_1 + \cdots + \langle v, e_n\rangle e_n$.

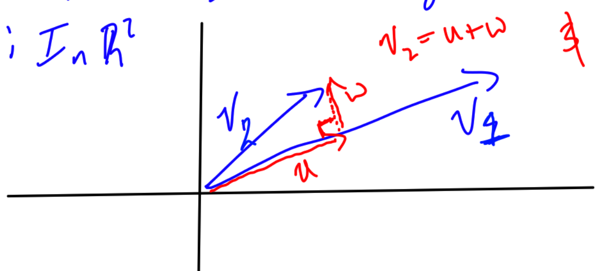
(b) Using (a), $\|v\|^2 = |\langle v, e_1\rangle|^2 + \cdots + |\langle v, e_n\rangle|^2$ via the first theorem of this section.

(c) Using part (a), we get

$$\begin{aligned}\langle u, v\rangle &= \langle \langle u, e_1\rangle e_1 + \cdots + \langle u, e_n\rangle e_n, \langle v, e_1\rangle e_1 + \cdots + \langle v, e_n\rangle e_n\rangle \\ &= \langle \langle u, e_1\rangle e_1, \langle v, e_1\rangle e_1 + \cdots + \langle v, e_n\rangle e_n\rangle + \cdots + \langle \langle u, e_n\rangle e_n, \langle v, e_1\rangle e_1 + \cdots + \langle v, e_n\rangle e_n\rangle \\ &= \langle u, e_1\rangle\left(\langle e_1, \langle v, e_1\rangle e_1\rangle + \cdots + \langle e_1, \langle v, e_n\rangle e_n\rangle\right) + \cdots + \langle u, e_n\rangle\left(\langle e_n, \langle v, e_1\rangle e_1\rangle + \cdots + \langle e_n, \langle v, e_n\rangle e_n\rangle\right) \\ &= \langle u, e_1\rangle\left(\overline{\langle v, e_1\rangle}\langle e_1, e_1\rangle + \cdots + \overline{\langle v, e_n\rangle}\langle e_1, e_n\rangle\right) + \cdots + \langle u, e_n\rangle\left(\overline{\langle v, e_1\rangle}\langle e_n, e_1\rangle + \cdots + \overline{\langle v, e_n\rangle}\langle e_n, e_n\rangle\right) \\ &= \langle u, e_1\rangle\overline{\langle v, e_1\rangle} + \cdots + \langle u, e_n\rangle\overline{\langle v, e_n\rangle}\end{aligned}$$

So, we are done!

Idea If we have a basis of an inner product space V, then how can we make that basis orthogonal in the easiest way? That is, visualize this: In $\mathbb{R}^2$



$\exists$ $\langle w, v_1\rangle = 0$

So, just take v_2 and remove the part of it that isn't orthogonal to v_1.

Then, normalize! i.e. replace v_1 w/ $\frac{v_1}{\|v_1\|}$ & w w/ $\frac{w}{\|w\|}$

6.32 *Gram–Schmidt procedure*

Suppose $v_1, \dots, v_m$ is a linearly independent list of vectors in V. Let $f_1 = v_1$. For $k = 2, \dots, m$, define f_k inductively by

$$f_k = v_k - \frac{\langle v_k, f_1\rangle}{\|f_1\|^2} f_1 - \cdots - \frac{\langle v_k, f_{k-1}\rangle}{\|f_{k-1}\|^2} f_{k-1}.$$

For each $k = 1, \dots, m$, let $e_k = \dfrac{f_k}{\|f_k\|}$. Then $e_1, \dots, e_m$ is an orthonormal list of vectors in V such that

$$\operatorname{span}(v_1, \dots, v_k) = \operatorname{span}(e_1, \dots, e_k)$$

for each $k = 1, \dots, m$.

$\boxed{\text{Proof}}$ (Sketch)

Do induction on k. If $k=1$, then, $f_1 = v_1$ & $e_1 = \frac{f_1}{\|f_1\|} = \frac{v_1}{\|v_1\|}$.

So, $\|e_1\| = \left\|\frac{v_1}{\|v_1\|}\right\| = \left|\frac{1}{\|v_1\|}\right| \|v_1\| = 1$, and $0 \neq e_1 \in \operatorname{span}(v_1)$ so $\operatorname{span}(e_1) = \operatorname{span}(v_1)$.

Then, & everything works up to $k-1$, i.e. that $e_1, \dots, e_{k-1}$ is an ON list s.t. $\operatorname{span}(v_1, \dots, v_{k-1}) = \operatorname{Span}(e_1, \dots, e_{k-1})$. Then, build e_k & show everything is still good.

$$f_k = v_k - \frac{\langle v_k, f_1\rangle}{\|f_1\|^2} f_1 - \cdots - \frac{\langle v_k, f_{k-1}\rangle}{\|f_{k-1}\|^2} f_{k-1}$$

$\leftarrow$ $v_k = -(\text{'' ''}) + f_k$ & v_k is then in $\operatorname{span}(e_1, \dots, e_k)$

$$e_k = \frac{f_k}{\|f_k\|}$$

So, $\langle e_k, e_j\rangle = \left\langle \frac{f_k}{\|f_k\|}, \frac{f_j}{\|f_j\|}\right\rangle$ (where $j < k$

$$= \frac{1}{\|f_k\|\|f_j\|}\langle f_k, f_j\rangle$$

$$= \frac{1}{\|f_k\|\|f_j\|}\left\langle v_k - \frac{\langle v_k, f_1\rangle}{\|f_1\|^2} f_1 - \cdots - \frac{\langle v_k, f_{k-1}\rangle}{\|f_{k-1}\|^2} f_{k-1}, f_j\right\rangle$$

$$= \frac{1}{\|f_k\|\|f_j\|}\left(\langle v_k, f_j\rangle - \frac{\langle v_k, f_1\rangle}{\|f_1\|^2}\langle f_1, f_j\rangle - \cdots - \frac{\langle v_k, f_{k-1}\rangle}{\|f_{k-1}\|^2}\langle f_{k-1}, f_j\rangle\right)$$

$$= \frac{1}{\|f_k\|\|f_j\|}\left(\langle v_k, f_j\rangle - \frac{\langle v_k, f_j\rangle}{\cancel{\|f_j\|^2}}\cancel{\langle f_j, f_j\rangle}\right)$$

$$= \frac{1}{\|f_k\|\|f_j\|}\left(\langle v_k, f_j\rangle - \langle v_k, f_j\rangle\right) = 0.$$

Then, show the span bit, which is not bad.

6.35 *existence of orthonormal basis*

Every finite-dimensional inner product space has an orthonormal basis.

Proof Take a basis & apply GS process.

6.36 *every orthonormal list extends to an orthonormal basis*

Suppose V is finite-dimensional. Then every orthonormal list of vectors in V can be extended to an orthonormal basis of V.

Proof Take the ON list, which is lin. indep., and extend to a basis. Then, apply GS to this basis. Only the elements added in the extension are changed, since the first part is already ON.

6.37 *upper-triangular matrix with respect to some orthonormal basis*

Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Then T has an upper-triangular matrix with respect to some orthonormal basis of V if and only if the minimal polynomial of T equals $(z - \lambda_1) \cdots (z - \lambda_m)$ for some $\lambda_1, \ldots, \lambda_m \in \mathbf{F}$.

Proof This is the same condition as what we had for getting an UT matrix of T w/ a regular basis, so just use the fact that we can orthonormalize this basis w/ GS & we are done (using $\operatorname{span}(v_1, \ldots, v_k) = \operatorname{span}(e_1, \ldots, e_k)$).

6.38 *Schur's theorem*

Every operator on a finite-dimensional complex inner product space has an upper-triangular matrix with respect to some orthonormal basis.