6A Inner Products and Norms

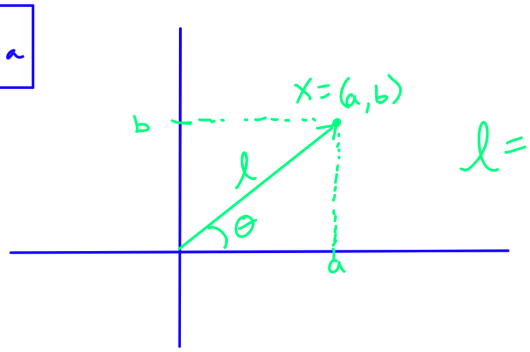
6.1 definition: *dot product*

For $x, y \in \mathbf{R}^n$, the *dot product* of x and y, denoted by $x \cdot y$, is defined by

$$x \cdot y = x_1 y_1 + \cdots + x_n y_n,$$

where $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$.

$\boxed{\text{Idea}}$



$\ell = \sqrt{b^2 + a^2} = \sqrt{bb + aa} = \sqrt{x \cdot x}$ ←

$\tan\theta = \frac{b}{a}$

$\theta = \tan^{-1}\left(\frac{b}{a}\right)$

6.2 definition: *inner product*

An *inner product* on V is a function that takes each ordered pair (u, v) of elements of V to a number $\langle u, v \rangle \in \mathbf{F}$ and has the following properties.

here, $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$

i.e. $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{F}$

positivity

$\langle v, v \rangle \geq 0$ for all $v \in V$.

definiteness

$\langle v, v \rangle = 0$ if and only if $v = 0$.

So, $\langle v, v \rangle > 0$ for $v \neq 0$

additivity in first slot

$\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$ for all $u, v, w \in V$.

homogeneity in first slot

$\langle \lambda u, v \rangle = \lambda \langle u, v \rangle$ for all $\lambda \in \mathbf{F}$ and all $u, v \in V$.

So, for fixed $w \in V$, the function $v \mapsto \langle v, w \rangle$ is a linear functional

conjugate symmetry ← Symmetry if $\mathbb{F} = \mathbb{R}$,

$\langle u, v \rangle = \overline{\langle v, u \rangle}$ for all $u, v \in V$. since $\overline{\langle v, u \rangle} = \langle v, u \rangle$ in $\mathbb{R}$

Recall: for $z = x + iy \in \mathbb{C}$, $\bar{z} = x - iy \in \mathbb{C}$ (i.e. throw a minus on the imaginary part)

6.4 definition: *inner product space*

An *inner product space* is a vector space V along with an inner product on V.

6.6 *basic properties of an inner product*

(a) For each fixed $v \in V$, the function that takes $u \in V$ to $\langle u, v\rangle$ is a linear map from V to $\mathbf{F}$.

(b) $\langle 0, v\rangle = 0$ for every $v \in V$.

(c) $\langle v, 0\rangle = 0$ for every $v \in V$.

(d) $\langle u, v + w\rangle = \langle u, v\rangle + \langle u, w\rangle$ for all $u, v, w \in V$.

(e) $\langle u, \lambda v\rangle = \overline{\lambda}\langle u, v\rangle$ for all $\lambda \in \mathbf{F}$ and all $u, v \in V$.

Proof

(a) Since $\langle u+w, v\rangle = \langle u,v\rangle + \langle w,v\rangle$ & $\langle \lambda u, v\rangle = \lambda\langle u,v\rangle$, we are done.

(b) This is true for any linear map, so we get (b) from (a)

(c) Using conjugate symmetry, $\langle v,0\rangle = \overline{\langle 0,v\rangle} = \overline{0} = 0$.

(d) $\langle u, v+w\rangle = \overline{\langle v+w, u\rangle} = \overline{\langle v,u\rangle + \langle w,u\rangle} = \overline{\langle v,u\rangle} + \overline{\langle w,u\rangle} = \langle u,v\rangle + \langle u,w\rangle$

(e) $\langle u, \lambda v\rangle = \overline{\langle \lambda v, u\rangle} = \overline{\lambda\langle v,u\rangle} = \overline{\lambda}\,\overline{\langle v,u\rangle} = \overline{\lambda}\langle u,v\rangle$

6.7 definition: *norm,* $\|v\|$

For $v \in V$, the *norm* of v, denoted by $\|v\|$, is defined by

$$\boxed{\|v\| = \sqrt{\langle v, v\rangle}.}$$

6.9 *basic properties of the norm*

Suppose $v \in V$.

(a) $\|v\| = 0$ if and only if $v = 0$.

(b) $\|\lambda v\| = |\lambda|\,\|v\|$ for all $\lambda \in \mathbf{F}$.

$\boxed{\text{Proof}}$ (a) $\|v\|=0 \iff \sqrt{\langle v,v\rangle} = 0$

$\iff \langle v,v\rangle = 0$

$\iff v = 0$

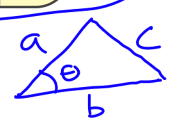
(b)
$$\begin{aligned}
\|\lambda v\| &= \sqrt{\langle \lambda v, \lambda v\rangle} \\
&= \sqrt{\lambda \langle v, \lambda v\rangle} \\
&= \sqrt{\lambda\, \overline{\langle \lambda v, v\rangle}} \\
&= \sqrt{\lambda \bar{\lambda}\, \overline{\langle v, v\rangle}} \\
&= \sqrt{\lambda \bar{\lambda} \langle v, v\rangle} \\
&= \sqrt{|\lambda|^2 \langle v, v\rangle} \\
&= |\lambda| \sqrt{\langle v, v\rangle} \\
&= |\lambda|\, \|v\|
\end{aligned}$$

$$\begin{aligned}
z &= x+iy \\
\bar{z} &= x-iy \\
z\bar{z} &= (x+iy)(x-iy) \\
&= x^2 + \cancel{iyx} - \cancel{iyx} - i^2y^2 \\
&= x^2+y^2 \\
&= |z|^2
\end{aligned}$$

6.10 definition: *orthogonal*

Two vectors $u, v \in V$ are called *orthogonal* if $\langle u, v\rangle = 0$.

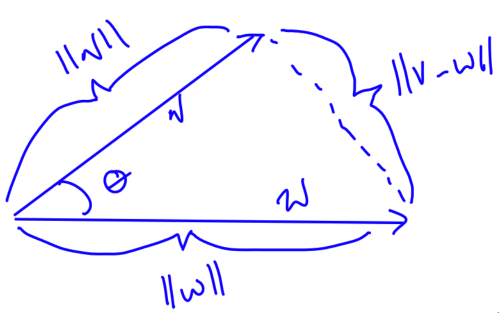
i.e. if they are at a right angle, using



$c^2 = a^2+b^2-2ab\cos\theta$

$\boxed{\text{Fact}}$ $\underbrace{\langle v,w\rangle = \|v\|\,\|w\|\cos\theta}$

using the law of cosines,



$$\|v-w\|^2 = \|w\|^2 + \|v\|^2 - 2\|v\|\,\|w\|\cos\theta$$

$$\langle v-w, v-w\rangle = \|w\|^2 + \|v\|^2 - 2\|v\|\,\|w\|\cos\theta$$

$$\cancel{\|v\|^2} + \cancel{\|w\|^2} - \cancel{2}\langle v,w\rangle = \cancel{\|w\|^2} + \cancel{\|v\|^2} - \cancel{2}\|v\|\,\|w\|\cos\theta$$

$$\underbrace{\langle v,w\rangle = \|v\|\,\|w\|\cos\theta}$$

Note: this calc assumes v, w are not scalar mult's of each other

6.11 *orthogonality and* 0

(a) 0 is orthogonal to every vector in V.

(b) 0 is the only vector in V that is orthogonal to itself.

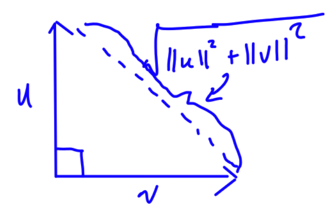
$\boxed{\text{Proof}}$ (a) $\langle 0, v\rangle = 0$ for all $v \in V$ by a previous result

(b) $\langle v, v\rangle = 0$ if & only if $v = 0$ by the definiteness of the inner product.

6.12 *Pythagorean theorem*

Suppose $u, v \in V$. If u and v are orthogonal, then

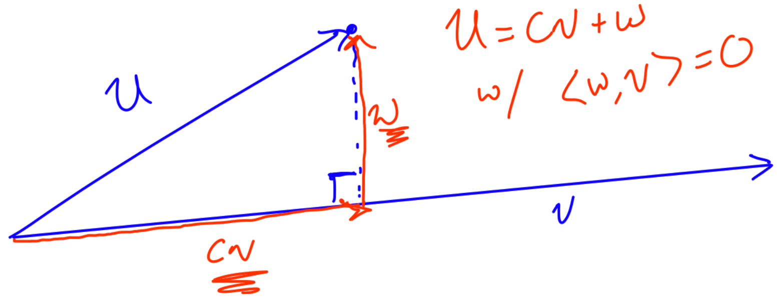
$$\|u + v\|^2 = \|u\|^2 + \|v\|^2.$$



$\boxed{\text{Proof}}$ Sp $\langle u, v\rangle = 0$. Then,

$$
\begin{aligned}
\|u+v\|^2 &= \langle u+v, u+v\rangle \\
&= \langle u, u+v\rangle + \langle v, u+v\rangle \\
&= \langle u, u\rangle + \langle u, v\rangle + \langle v, u\rangle + \langle v, v\rangle \\
&= \|u\|^2 + \langle u, v\rangle + \overline{\langle u, v\rangle} + \|v\|^2 \\
&= \|u\|^2 + 0 + \bar{0} + \|v\|^2 \\
&= \|u\|^2 + \|v\|^2
\end{aligned}
$$

$\boxed{\text{Idea}}$



$u = cv + w$
w/ $\langle w, v\rangle = 0$

6.13 *an orthogonal decomposition*

Suppose $u, v \in V$, with $v \neq 0$. Set $c = \dfrac{\langle u, v\rangle}{\|v\|^2}$ and $w = u - \dfrac{\langle u, v\rangle}{\|v\|^2} v$. Then

$$u = cv + w \quad \text{and} \quad \langle w, v\rangle = 0.$$

$\boxed{\text{Proof}}$ $cv + w = \underbrace{\dfrac{\langle u,v\rangle}{\|v\|^2} v} + u - \underbrace{\dfrac{\langle u,v\rangle}{\|v\|^2} v} = u$

$$\begin{aligned}\langle w, v\rangle = \left\langle u - \frac{\langle u,v\rangle}{\|v\|^2} v, v\right\rangle &= \langle u, v\rangle - \left\langle \frac{\langle u,v\rangle}{\|v\|^2} v, v\right\rangle \\ &= \langle u, v\rangle - \frac{\langle u,v\rangle}{\|v\|^2}\langle v, v\rangle \\ &= \langle u, v\rangle - \langle u, v\rangle \frac{\|v\|^2}{\|v\|^2} = 0\end{aligned}$$

6.14 *Cauchy–Schwarz inequality* (Cauchy–Bunyakovsky–Schwarz inequality)

Suppose $u, v \in V$. Then

$$|\langle u, v\rangle| \leq \|u\|\,\|v\|.$$

This inequality is an equality if and only if one of u, v is a scalar multiple of the other.

$\boxed{\text{Proof}}$ (assume $v \neq 0$) If $u = 0$, then both sides of the inequality are zero, so it holds. If not, then $u = cv + w$ where $\langle w, v\rangle = 0$. Using the pythagorean theorem, we get

$$\begin{aligned}\|u\|^2 = \|cv + w\|^2 &= \|cv\|^2 + \|w\|^2 \quad (\text{since } \langle cv, w\rangle = c\langle v, w\rangle = 0) \\ &= \left\|\frac{\langle u,v\rangle}{\|v\|^2} v\right\|^2 + \|w\|^2 \\ &= \frac{|\langle u,v\rangle|^2}{\|v\|^4}\|v\|^2 + \|w\|^2 \\ &= \frac{|\langle u,v\rangle|^2}{\|v\|^2} + \|w\|^2 \\ &\geq \frac{|\langle u,v\rangle|^2}{\|v\|^2}\end{aligned}$$

equality iff $w = 0$ & $u = cv$

So, $\|u\|^2 \geq \dfrac{|\langle u,v\rangle|^2}{\|v\|^2}$ & $\|u\|^2\|v\|^2 \geq |\langle u,v\rangle|^2$, or $|\langle u,v\rangle| \leq \|u\|\,\|v\|$.

6.17 *triangle inequality*

Suppose $u, v \in V$. Then

$$\|u + v\| \leq \|u\| + \|v\|.$$

This inequality is an equality if and only if one of u, v is a nonnegative real multiple of the other.

$\boxed{\text{Proof}}$

$$\begin{aligned} \|u+v\|^2 &= \langle u+v, u+v\rangle \\ &= \langle u,u\rangle + \langle u,v\rangle + \langle v,u\rangle + \langle v,v\rangle \\ &= \|u\|^2 + \|v\|^2 + \langle u,v\rangle + \overline{\langle u,v\rangle} \\ &= \|u\|^2 + \|v\|^2 + 2\,\mathrm{Re}(\langle u,v\rangle) \\ &\leq \|u\|^2 + \|v\|^2 + 2\,|\langle v,u\rangle| \\ &\leq \|u\|^2 + \|v\|^2 + 2\,\|v\|\,\|u\| \quad (\text{via CS}) \\ &= (\|u\| + \|v\|)^2 \end{aligned}$$

$z = x + iy$

$\bar{z} = x - iy$

$z + \bar{z} = 2x$

i.e. $z + \bar{z} = 2\,\mathrm{Re}(z)$

$|z| = x^2 + y^2$

$\geq x^2$

So, $2|z| \geq 2\,\mathrm{Re}(z)^2 \geq 2\,\mathrm{Re}(z)$

So, $\|u+v\|^2 \leq (\|u\| + \|v\|)^2$ & $\|u+v\| \leq \|u\| + \|v\|$

6.21 *parallelogram equality*

Suppose $u, v \in V$. Then

$$\|u + v\|^2 + \|u - v\|^2 = 2(\|u\|^2 + \|v\|^2).$$

$\boxed{\text{Proof}}$

$$\begin{aligned} \|u+v\|^2 + \|u-v\|^2 &= \langle u+v, u+v\rangle + \langle u-v, u-v\rangle \\ &= \langle u,u\rangle + \cancel{\langle u,v\rangle} + \cancel{\langle v,u\rangle} + \langle v,v\rangle \\ &\quad + \langle u,u\rangle - \cancel{\langle u,v\rangle} - \cancel{\langle v,u\rangle} + \langle v,v\rangle \\ &= 2\langle u,u\rangle + 2\langle v,v\rangle \\ &= 2(\|u\|^2 + \|v\|^2) \end{aligned}$$