# 5E Commuting Operators

i.e. $S, T \in \mathcal{L}(V)$

**5.71 definition:** *commute*

- Two operators $S$ and $T$ on the same vector space *commute* if $ST = TS$.
- Two square matrices $A$ and $B$ of the same size *commute* if $AB = BA$.

**5.74** *commuting operators correspond to commuting matrices*

Suppose $S, T \in \mathcal{L}(V)$ and $v_1, \ldots, v_n$ is a basis of $V$. Then $S$ and $T$ commute if and only if $\mathcal{M}(S, (v_1, \ldots, v_n))$ and $\mathcal{M}(T, (v_1, \ldots, v_n))$ commute.

$\boxed{\text{Proof}}$

$$\begin{aligned} S \,\&\, T \text{ commute} &\iff ST = TS \\ &\iff \mathcal{M}(ST) = \mathcal{M}(TS) \\ &\iff \mathcal{M}(S)\mathcal{M}(T) = \mathcal{M}(T)\mathcal{M}(S) \\ &\iff \mathcal{M}(S) \,\&\, \mathcal{M}(T) \text{ commute} \end{aligned}$$

**5.75** *eigenspace is invariant under commuting operator*

Suppose $S, T \in \mathcal{L}(V)$ commute and $\lambda \in \mathbf{F}$. Then $E(\lambda, S)$ is invariant under $T$.

$\boxed{\text{Proof}}$ We want to show that $v \in E(\lambda, S)$ implies $Tv \in E(\lambda, S)$.

$\$$ $v \in E(\lambda, S)$. Then,

$$S(Tv) = TSv = T(Sv) = T(\lambda v) = (\lambda Tv) \in E(\lambda, S)$$

i.e. $STv = \lambda Tv$, so $(S - \lambda I)(Tv) = 0$ $\$$ $Tv \in E(\lambda, S)$.

Notice: $E(\lambda, T)$ is also invariant under $S$ (just swap $S \$ T$ in this proof).

**5.76** *simultaneous diagonalizablity $\iff$ commutativity*

Two diagonalizable operators on the same vector space have diagonal matrices with respect to the same basis if and only if the two operators commute.

$\boxed{\text{Proof}}$ Suppose $S, T \in \mathcal{L}(V)$ have diagonal matrices w.r.t. a basis $v_1, \dots, v_n$ of $V$. Then, $\mathcal{M}(S)\mathcal{M}(T) = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}\begin{bmatrix} \eta_1 & & 0 \\ & \ddots & \\ 0 & & \eta_n \end{bmatrix}$

$$= \begin{bmatrix} \lambda_1\eta_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n\eta_n \end{bmatrix}$$

$$= \begin{bmatrix} \eta_1\lambda_1 & & 0 \\ & \ddots & \\ 0 & & \eta_n\lambda_n \end{bmatrix}$$

$$= \begin{bmatrix} \eta_1 & & 0 \\ & \ddots & \\ 0 & & \eta_n \end{bmatrix}\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} = \mathcal{M}(T)\mathcal{M}(S)$$ So, $S$ & $T$ commute.

In the other direction, suppose $S, T \in \mathcal{L}(V)$ are diagonalizable commuting operators. Let $\lambda_1, \dots, \lambda_m$ denote the distinct eigenvalues of $S$. Then, we have

$$V = E(\lambda_1, S) \oplus \cdots \oplus E(\lambda_m, S) \quad (\text{since } S \text{ is diagonalizable})$$

Since $E(\lambda_k, S)$ is invariant under $T$ for $k \in \{1, \dots, m\}$, $T|_{E(\lambda_k,S)} \in \mathcal{L}(E(\lambda_k, S))$ & is also diagonalizable (since $T$ is). So, $E(\lambda_k, S)$ has a basis of eigenvectors of $T|_{E(\lambda_k,S)}$ & hence of $T$. Combining bases for each, we get a basis of $V$ that consists of eigenvectors of $T$ which are also eigenvectors of $S$. Hence, the matrices of $S$ & $T$ w.r.t. this basis are both diagonal.

**5.78** *common eigenvector for commuting operators*

Every pair of commuting operators on a finite-dimensional nonzero complex vector space has a common eigenvector.

$\boxed{\text{Proof}}$ Suppose $V$ is a nonzero complex finite-dim. VS. & that $S, T \in \mathcal{L}(V)$ commute. Let $\lambda$ be an eigenvalue of $S$ (which exists since $V$ is complex & finite-dim.). So, $E(\lambda, S) \neq \{0\}$ is invariant under $T$ & therefore $T|_{E(\lambda,S)} \in \mathcal{L}(E(\lambda, S))$.

So, $T|_{E(\lambda,S)}$ has an eigenvalue (for the same reason $S$ does) & must therefore have an eigenvector. But, this eigenvector is in $E(\lambda, S)$, so it is also an eigenvector of $S$.

**5.80** *commuting operators are simultaneously upper triangularizable*

Suppose $V$ is a finite-dimensional complex vector space and $S, T$ are commuting operators on $V$. Then there is a basis of $V$ with respect to which both $S$ and $T$ have upper-triangular matrices.

$\boxed{\text{Proof}}$ We do induction on $n = \dim V$. Base case: For $n=1$, every matrix of any operator is diagonal, so we are done.

Inductive Step: S the result is true for all values less than $n = \dim V$. Let $S, T \in \mathcal{L}(V)$ be commuting operators. Then, by our previous result, $S$ & $T$ have a common eigenvector. Let $v_1$ be such an eigenvector of both $S$ & $T$. Then, $\underline{Sv_1 \in \operatorname{Span}(v_1)}$ & $\underline{Tv_1 \in \operatorname{span}(v_1)}$.

So, let $W \leq V$ s.t. $V = \operatorname{span}(v_1) \oplus W$. Define $P \in \mathcal{L}(V, W)$ by $P(v) = P(av_1 + w) = w$. Then, we will show $PS|_W, PT|_W \in \mathcal{L}(W)$ (here, $S|_W \in \mathcal{L}(W,V)$ & $T|_W \in \mathcal{L}(W,V)$) commute.

$$\left((PS|_W)(PT|_W)w\right) = (PS|_W)(\underbrace{Tw - av_1}) = P(STw - Sav_1) = P(STw - \tilde{a}v_1) = PSTw = PTSw = (PT|_W)(PS|_W)w$$

where $Tw = \tilde{w} + av_1$ for $\tilde{w} \in W$

So, $PS|_W$ & $PT|_W$ are commuting operators on $W$ & $\dim W = \dim V - 1 = n-1 < n$.

Therefore, the inductive hypothesis applies & both can be simultaneously upper-triangularized. Hence, taking $v_2, \ldots, v_n$ to be the basis of $W$ doing this, $v_1, \ldots, v_n$ simultaneously upper-triangularizes $S$ & $T$, since $Sv_k \in \operatorname{Span}(v_1, \ldots, v_k)$ & $Tv_k \in \operatorname{Span}(v_1, \ldots, v_k)$.

**5.81** *eigenvalues of sum and product of commuting operators*

Suppose $V$ is a finite-dimensional complex vector space and $S, T$ are commuting operators on $V$. Then

- every eigenvalue of $S + T$ is an eigenvalue of $S$ plus an eigenvalue of $T$,
- every eigenvalue of $ST$ is an eigenvalue of $S$ times an eigenvalue of $T$.

$\boxed{\text{Proof}}$ First, note that if $A, B$ are upper-triangular matrices, then so is $A+B$ & $AB$. Moreover, the diagonals of $A+B$ and $AB$ are the sums (resp. products) of the diagonal entries on $A$ & $B$. That is,

$$A+B = \begin{bmatrix} a_{1,1} & \cdots & a_{1,n} \\ & \ddots & \vdots \\ 0 & & a_{n,n} \end{bmatrix} + \begin{bmatrix} b_{1,1} & \cdots & b_{1,n} \\ & \ddots & \vdots \\ 0 & & b_{n,n} \end{bmatrix} = \begin{bmatrix} (a_{1,1}+b_{1,1}) & \cdots & (a_{1,n}+b_{1,n}) \\ & \ddots & \\ 0 & & (a_{n,n}+b_{n,n}) \end{bmatrix}$$

$$AB = \begin{bmatrix} a_{1,1} & \cdots & a_{1,n} \\ & \ddots & \vdots \\ 0 & & a_{n,n} \end{bmatrix} \begin{bmatrix} b_{1,1} & \cdots & b_{1,n} \\ & \ddots & \vdots \\ 0 & & b_{n,n} \end{bmatrix} = \begin{bmatrix} (a_{1,1}b_{1,1}) & & * \\ & \ddots & \\ 0 & & (a_{n,n}b_{n,n}) \end{bmatrix}$$

So, since there exists a basis of $V$ w.r.t. which both $S$ & $T$ have upper triangular matrices, and since the entries on the diagonal of an UT matrix of an operator are the eigenvalues of the operator, we are done!