

Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Then T is diagonalizable if and only if the minimal polynomial of T equals $(z - \lambda_1) \cdots (z - \lambda_m)$ for some list of distinct numbers $\lambda_1, \dots, \lambda_m \in \mathbb{F}$.

Proof First, $\S$ T is diagonalizable. Then, V has a basis of e-vec's of T . Let $v_1, \dots, v_n$ be such a basis & let $\lambda_1, \dots, \lambda_m$ be the distinct eigenvalues of T . Then, for each v_k , there exist λ_j s.t. $(T - \lambda_j I)v_k = 0$. So, $(T - \lambda_1 I) \cdots (T - \lambda_m I) = 0$ since it is zero on a basis (as $p(T)q(T) = q(T)p(T)$ for polys p, q). Hence, the min. poly. of T is a poly. multiple of $(z - \lambda_1) \cdots (z - \lambda_m)$ & is in fact the minimal poly of T , since any divisor of it leaves some eigenspace out of its null space.

Now, in the other direction, $\S$ the min. poly. of T is $(z - \lambda_1) \cdots (z - \lambda_m)$ for some list of distinct $\lambda_1, \dots, \lambda_m \in \mathbb{F}$.

We will do induction on m .

Base case: Let $m=1$. Then, $T - \lambda_1 I = 0$ & $T = \lambda_1 I$ so T is a scalar multiple of the identity & $M(T)$ is diagonal w.r.t. any basis of V (convince yourself of this by recalling the def'n of $M(T)$).

Inductive Step: $\S$ our desired result is true for all values less than $m > 1$, and let the min. poly. of T be $(z - \lambda_1) \cdots (z - \lambda_m)$ for some list of distinct $\lambda_1, \dots, \lambda_m \in \mathbb{F}$. We will show that $\text{range}(T - \lambda_m I)$ has a basis of e-vectors of T & that $V = \text{range}(T - \lambda_m I) \oplus \text{null}(T - \lambda_m I)$, showing then that any extension of this basis of e-vectors for $\text{range}(T - \lambda_m I)$ adds only more eigenvectors of T , and hence that V has a basis of e-vec's of T & is therefore diagonalizable.

Proof (continued)

First, note that $\text{range}(T - \lambda_m I)$ is an invariant subspace of T (since $\text{range } p(T)$ is invariant under T for any poly p).

So, $T|_{\text{range}(T - \lambda_m I)} \in \mathcal{L}(\text{range}(T - \lambda_m I))$. Now, if $u \in \text{range}(T - \lambda_m I)$, then $\exists v \in V$ st. $(T - \lambda_m I)v = u$. So,

$$(T - \lambda_1 I) \cdots (T - \lambda_{m-1} I)u = (T - \lambda_1 I) \cdots (T - \lambda_{m-1} I)(T - \lambda_m I)v = 0.$$

Hence, $(z - \lambda_1) \cdots (z - \lambda_{m-1})$ is a poly. mult. of the min. poly. of $T|_{\text{range}(T - \lambda_m I)}$. By our inductive hypothesis, this means that $T|_{\text{range}(T - \lambda_m I)}$ is diagonalizable & $\text{range}(T - \lambda_m I)$ has a basis of e-vec's of $T|_{\text{range}(T - \lambda_m I)}$ (and therefore also of T).

Now, to show $V = \text{range}(T - \lambda_m I) \oplus \text{null}(T - \lambda_m I)$, let $u \in \text{range}(T - \lambda_m I) \cap \text{null}(T - \lambda_m I)$. we show $u = 0$ to get $\text{range}(T - \lambda_m I) \cap \text{null}(T - \lambda_m I) = \{0\}$ & get $V = \underline{\text{range}(T - \lambda_m I)} \oplus \underline{\text{null}(T - \lambda_m I)}$. Since $u \in \text{null}(T - \lambda_m I)$, $Tu = \lambda_m u$. By our argument before,

$$0 = (T - \lambda_1 I) \cdots (T - \lambda_{m-1} I)u = (\lambda_m - \lambda_1) \cdots (\lambda_m - \lambda_{m-1})u$$

Since $\lambda_m - \lambda_k \neq 0$ for $k \neq m$ (as $\lambda_1, \dots, \lambda_m$ are distinct), this means $u = 0$. So, we are done!

(Wow, what a proof!)

5.65 restriction of diagonalizable operator to invariant subspace

Suppose $T \in \mathcal{L}(V)$ is diagonalizable and U is a subspace of V that is invariant under T . Then $T|_U$ is a diagonalizable operator on U .

Proof If T is diagonalizable, then the min. poly. of T is of the form $(z - \lambda_1) \cdots (z - \lambda_n)$ for some list of distinct $\lambda_1, \dots, \lambda_n \in \mathbb{F}$. So, $T|_U$ has a min. poly. of this form as well, since its min. poly. divides the min. poly. of T . Hence, it is also diagonalizable.

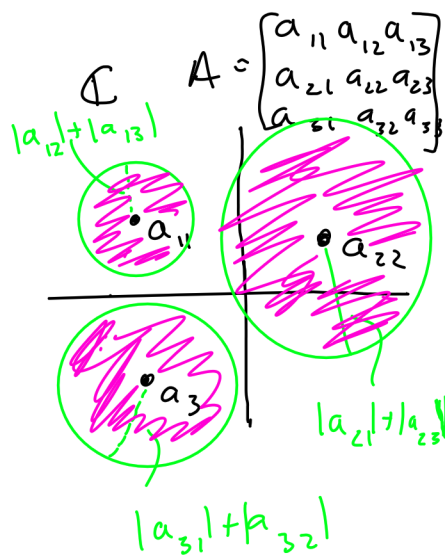
5.66 definition: Gershgorin disks

Suppose $T \in \mathcal{L}(V)$ and $v_1, \dots, v_n$ is a basis of V . Let A denote the matrix of T with respect to this basis. A Gershgorin disk of T with respect to the basis $v_1, \dots, v_n$ is a set of the form

$$\{z \in \mathbb{F} : |z - A_{j,j}| \leq \sum_{\substack{k=1 \\ k \neq j}}^n |A_{j,k}|\},$$

where $j \in \{1, \dots, n\}$.

Here, $\mathbb{F} = \mathbb{C}$ or $\mathbb{F} = \mathbb{R}$



5.67 Gershgorin disk theorem

Suppose $T \in \mathcal{L}(V)$ and $v_1, \dots, v_n$ is a basis of V . Then each eigenvalue of T is contained in some Gershgorin disk of T with respect to the basis $v_1, \dots, v_n$.

→ All the e-values live in the green part

Proof Let $\lambda \in \mathbb{F}$ be an eigenvalue of T . Let $w \in V$ be a corresponding e-vec. Since $v_1, \dots, v_n$ is a basis of V , $\exists c_1, \dots, c_n \in \mathbb{F}$ st. $w = c_1 v_1 + \dots + c_n v_n$. Now, apply T to both sides, writing $A = M(T)$ wrt. the basis $v_1, \dots, v_n$:

$$\lambda w = c_1 T v_1 + \dots + c_n T v_n = \sum_{k=1}^n c_k \sum_{j=1}^n A_{j,k} v_j = \sum_{k=1}^n \sum_{j=1}^n c_k A_{j,k} v_j = \sum_{j=1}^n \left(\sum_{k=1}^n c_k A_{j,k} \right) v_j.$$

so, $\sum_{k=1}^n c_k A_{j,k} = \lambda c_j$. Hence, $\lambda c_j - A_{j,j} c_j = \sum_{\substack{k=1 \\ k \neq j}}^n c_k A_{j,k}$ & $|\lambda - A_{j,j}| \leq \sum_{\substack{k=1 \\ k \neq j}}^n \frac{|c_k|}{|c_j|} |A_{j,k}|$.

so, if $|c_j|$ is the max of $|c_1|, \dots, |c_n|$, $\frac{|c_k|}{|c_j|} \leq 1$ & we're done!